A CODING SCHEME FOR THE ADDITIVE WHITE GAUSSIAN NOISE MULTIPLE ACCESS CHANNEL WITH SEMI-FEEDBACK

Dr.ir.F.M.J.Willems

Afdeling der Elektrotechniek
Technische Hogeschool Eindhoven.

Prof.dr.E.C.van der Meulen

Departement Wiskunde
Katholieke Universiteit Leuven.

Prof.dr.ir. J.P.M.Schalkwijk

Afdeling der Elektrotechniek
Technische Hogeschool Eindhoven.

Motivated by Ozarow's doctoral dissertation, a semi-feedback scheme is presented for the additive white Gaussian noise multiple access channel, which yields rate-pairs outside the non-feedback capacity-region.

1. INTRODUCTION

In this investigation we study the implications of the Schalkwijk [5] scheme for the discrete-time additive white Gaussian noise (AWGN) multiple access channel (MAC) with instantaneous feedback to only one encoder. In order to demonstrate the Schalkwijk feedback scheme we first consider a one-way communication situation (figure 1) where a real valued message θ is sent from transmitter to receiver. The maximum average transmitter power is P

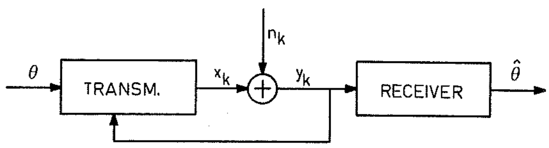

Fig. 1. Discrete-time AWGN one-way channel with feedback.

and the noise variance is N. We assume that θ can take on any $\|\theta\|$ equally spaced values in the interval $[-0.5,0.5]$. Each θ has probability $1/\|\theta\|$. Now in the first of a block of n transmissions we send:

$$m_1=\theta \text{ hence } a_1 \stackrel{\Delta}{=} \text{var}\{m_1\}=\text{var}\{\theta\}. \tag{1}$$

The first channel input is:

$$x_1=S_1 m_1 \text{ where } S_1=\frac{\sqrt{P}}{a_1}. \tag{2}$$

The first output is

$$y_1=x_1+n_1=S_1 m_1+n_1. \tag{3}$$

Here n_1 stands for the noise in transmission 1. From the first output the receiver calculates the maximum-likelihood estimate of m_1:

$$\hat{m}_1=G_1 y_1=m_1+G_1 n_1 \text{ where } G_1=\frac{1}{S_1} \tag{4}$$

and sets the first estimate of θ

$$\hat{\theta}_1=\hat{m}_1=\theta+G_1 n_1. \tag{5}$$

The transmitter is also able to compute the receiver's estimate since it has the output available via the feedback link. Now from this estimate the transmitter calculates the message for the second transmission:

$$m_2=\theta-\hat{\theta}_1=-G_1 n_1. \tag{6}$$

Note that m_2 is Gaussian with zero mean.
Now consider the k-th transmission ($2 \le k \le n$). Message m_k with variance a_k has to be communicated; the transmitter sets the channel input:

$$x_k=S_k m_k \text{ where } S_k=\frac{\sqrt{P}}{a_k}. \tag{7}$$

The receiver now computes the best mean-square-error estimate of m_k:

$$m_k=G_k y_k \text{ where } G_k=\frac{1}{S_k}\cdot\frac{P}{P+N} \tag{8}$$

and sets the k-th estimate of θ:

$$\hat{\theta}_k=\hat{\theta}_{k-1}+\hat{m}_k. \tag{9}$$

The transmitter calculates the message to be sent in the next transmission:

$$m_{k+1}=\theta-\hat{\theta}_k=\theta-\hat{\theta}_{k-1}-\hat{m}_k=m_k-\hat{m}_k. \tag{10}$$

One easily checks that because m_k is Gaussian with zero mean m_{k+1} also is Gaussian with zero mean and that:

$$\frac{a_{k+1}}{a_k}=\frac{N}{P+N}. \tag{11}$$

After n transmissions the probability of error:

$$P_e \le \Pr\left[\,|\hat{\theta}_n-\theta| > \frac{1}{2(||\theta||-1)}\,\right]. \qquad (12)$$

From the foregoing it follows that:

$$\mathrm{var}\{(\hat{\theta}_n-\theta)\}=a_1\left(\frac{N}{P}\right)\left(\frac{N}{P+N}\right)^{n-1}. \qquad (13)$$

Therefore P_e decreases doubly exponential with the block-length n if:

$$R=\frac{\ln(||\theta||)}{n}<\frac{1}{2}\ln\left(\frac{P+N}{N}\right) \text{ nats/transmission.} \qquad (14)$$

It follows that for the one-way channel using the Schalkwijk feedback scheme we can transmit reliably at rates up to capacity.

In the following section we will investigate the performance of the Schalkwijk feedback-scheme for the MAC with semi-feedback.
It should be remarked that the work of Ozarow [4] can be seen as the instigation of this research.

2. THE AWGN MAC WITH SEMI-FEEDBACK

Ozorow [4] in his doctoral dissertation investigated the AWGN MAC with feedback to both transmitters and determined the capacity region for this MAC suitably applying the constructive Schalkwijk scheme. This region improved upon the Cover-Leung [1] region in this case. Dueck [2] investigated the problem of semi-feedback for discrete multi-user channels. He found that for the binary erasure MAC one can achieve with semi-feedback the same rate-point as Gaarder and Wolf [3] found using complete feedback. In this section we will study the implications of the Schalkwijk scheme for AWGN MAC's in the case of semi-feedback. It is found that with a semi-constructive and semi-feedback scheme one can generally improve upon the classical capacity-region without feedback.

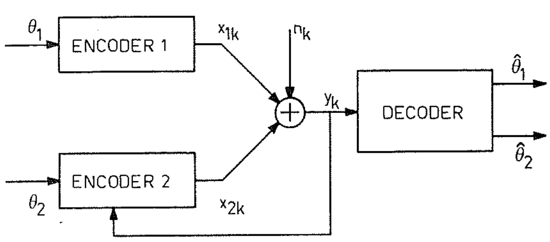

Fig.2. Discrete-time AWGN MAC with semi-feedback.

Consider the communication situation of figure 2. The variance of the noise is N and the transmitter powers are P_1 and P_2 for encoder 1 and encoder 2 respectively. Now encoder 1 transmits messages $\theta_1\in\{1,\ldots,\exp(nR_1)\}$ in a block of n transmissions. All messages have the same probability. In order to transmit these messages encoder 1 uses a somewhat special random code. First $\exp(nR_1)$ codewords m_1^{0n} are generated according to:

$$p(m_1^{0n})=\prod_{k=0,n}p(m_{1k}) \text{ where } p(m_{1k})=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{m_{1k}^2}{2}\right). \qquad (15)$$

From these codewords m_1^{0n} the codewords x_1^n are constructed in the following way $(1\le k\le n)$:

$$x_{1k}=\sqrt{\frac{P_1}{1+\gamma^2}}\,(m_{1k}-\gamma m_{1k-1}),\ \gamma\ge 0. \qquad (16)$$

Encoder 2 transmits in a Schalkwijk way the real valued message θ_2 to the decoder also in a block of n transmissions. Assume that $\overline{m_{10}m_{21}}=\rho_1\mathrm{var}\{m_{21}\}$ and that m_{21} is Gaussian with zero mean. This can be accomplished with methods similar to the techniques Ozarow uses to initialize his communication process. Suppose after k-1 channel uses the decoder has the following estimate of θ_2:

$$\hat{\theta}_{2k-1}=\theta_2-m_{2k} \text{ and } \mathrm{var}\{m_{2k}\}=a_k. \qquad (17)$$

Encoder 2 because of the feedback link also knowing this estimate then sets:

$$x_{2k}=S_{2k}m_{2k} \text{ where } S_{2k}=\sqrt{\frac{P_2}{a_k}}. \qquad (18)$$

Upon receiving y_k the decoder computes the best mean-square-error estimate of m_{2k} and then forms:

$$\hat{\theta}_{2k}=\hat{\theta}_{2k-1}+\hat{m}_{2k}=\hat{\theta}_{2k-1}+G_ky_k$$
$$=\theta_2-m_{2k}+G_ky_k\ . \qquad (19)$$

Hence by (17):

$$m_{2k+1}=m_{2k}-G_ky_k=m_{2k}-\hat{m}_{2k}. \qquad (20)$$

We will now calculate G_k such that $\hat{m}_{2k}$ is the best mean square error estimate of m_{2k}. First set:

$$S_1=\sqrt{\frac{P_1}{1+\gamma^2}}\ . \qquad (21)$$

Then:

$$y_k=x_{1k}+x_{2k}+n_k$$
$$=S_1(m_{1k}-\gamma m_{1k-1})+S_{2k}m_{2k}+n_k. \qquad (22)$$

We know that:

$$\overline{m_{1k-1}m_{1k}}=\overline{m_{1k}m_{2k}}=0. \qquad (23)$$

With $\hat{m}_{2k}=G_ky_k$ it follows that:

$$\overline{(m_{2k}-\hat{m}_{2k})^2}=G_k^2(S_1^2\overline{m_{1k}^2}+\gamma^2S_1^2\overline{m_{1k-1}^2}+S_{2k}^2\overline{m_{2k}^2}-2\gamma S_1S_{2k}\overline{m_{1k-1}m_{2k}}+\overline{n_k^2})$$

$$-2G_k(S_{2k}\overline{m_{2k}^2}-\gamma S_1\overline{m_{1k-1}m_{2k}})+\overline{m_{2k}^2}. \qquad (24)$$

It is easy to see that (24) is minimized by:

$$G_k=\frac{1}{S_{2k}}\cdot\frac{S_{2k}^2\overline{m_{2k}^2}-\gamma S_1S_{2k}\overline{m_{1k-1}m_{2k}}}{S_1^2\overline{m_{1k}^2}+\gamma^2S_1^2\overline{m_{1k+1}^2}+S_{2k}^2\overline{m_{2k}^2}-2\gamma S_1S_{2k}\overline{m_{1k-1}m_{2k}}+\overline{n_k^2}} \qquad (25)$$

With G_k as above:

$$\frac{a_{k+1}}{a_k}=\frac{\overline{(m_{2k}-\hat{m}_{2k})^2}}{\overline{m_{2k}^2}}$$

$$=\frac{S_1^2\overline{m_{1k}^2}+\overline{n_k^2}+\gamma^2S_1^2\overline{m_{1k-1}^2}\left(1-\frac{\overline{m_{1k-1}m_{2k}}^2}{\overline{m_{1k-1}^2}\cdot\overline{m_{2k}^2}}\right)}{S_1^2\overline{m_{1k}^2}+\gamma^2S_1^2\overline{m_{1k-1}^2}+S_{2k}^2\overline{m_{2k}^2}-2\gamma S_1S_{2k}\overline{m_{1k-1}m_{2k}}+\overline{n_k^2}}. \qquad (26)$$

Now substitute:

$$\overline{m_{1k}^2}=\overline{m_{1k-1}^2}=1, \qquad S_1=\sqrt{\frac{P_1}{1+\gamma^2}},$$

$$\overline{m_{2k}^2}=a_k, \qquad S_{2k}=\sqrt{\frac{P_2}{a_k}},$$

$$\overline{m_{1k-1}m_{2k}}=\rho_k\sqrt{a_k}. \qquad (27)$$

Then

$$G_k=\sqrt{\frac{a_k}{P_2}}\,\frac{P_2-\rho_k\sqrt{\frac{\gamma^2}{1+\gamma^2}}\sqrt{P_1P_2}}{P_1+P_2-2\rho_k\sqrt{\frac{\gamma^2}{1+\gamma^2}}\sqrt{P_1P_2}+N} \qquad (28)$$

and

$$\frac{a_{k+1}}{a_k}=\frac{N+P_1(1-\rho_k^2\frac{\gamma^2}{1+\gamma^2})}{P_1+P_2-2\rho_k\sqrt{\frac{\gamma^2}{1+\gamma^2}}\sqrt{P_1P_2}+N}. \qquad (29)$$

Now define:

$$\rho_k^*=-\rho_k\sqrt{\frac{\gamma^2}{1+\gamma^2}}, \qquad (30)$$

then:

$$G_k=\sqrt{\frac{a_k}{P_2}}\cdot\frac{P_2+\rho_k^*\sqrt{P_1P_2}}{P_1+P_2+2\rho_k^*\sqrt{P_1P_2}+N} \qquad (31)$$

and

$$\frac{a_{k+1}}{a_k}=\frac{N+P_1(1-\rho_k^{*2})}{P_1+P_2+2\rho_k^*\sqrt{P_1P_2}+N} \qquad (32)$$

Now what about ρ_{k+1}?

$$\rho_{k+1}=\frac{\overline{m_{1k}m_{2k+1}}}{\sqrt{a_{k+1}}} \qquad (33)$$

where:

$$\overline{m_{1k}m_{2k+1}}=\overline{m_{1k}(m_{2k}-G_ky_k)}$$

$$=\overline{m_{1k}(m_{2k}-G_k(S_1(m_{1k}-\gamma m_{1k-1})+S_{2k}m_{2k}+n_k))}$$

$$=-G_kS_1\overline{m_{1k}^2}=-G_kS_1. \qquad (34)$$

Hence:

$$\rho_{k+1}=\frac{-G_kS_1}{\sqrt{a_{k+1}}}=-\sqrt{\frac{a_k}{a_{k+1}}}\cdot\sqrt{\frac{P_1}{P_2(1+\gamma^2)}}\cdot\frac{P_1+\rho_k^*\sqrt{P_1P_2}}{P_1+P_2+2\rho_k^*\sqrt{P_1P_2}+N}$$

or:

$$\rho_{k+1}^{*2}=\frac{\gamma^2}{(1+\gamma^2)^2}\cdot\frac{P_1}{P_2}\cdot\frac{(P_2+\rho_k^*\sqrt{P_1P_2})^2}{(N+P_1(1-\rho_k^{*2}))(P_1+P_2+2\rho_k^*\sqrt{P_1P_2}+N)}. \qquad (35)$$

Suppose $\rho_{k+1}^*=\rho_k^*=\rho^*\geq 0$. Now what is γ^2 as a function of ρ^*? Define:

$$C(\rho^*)=\frac{P_2}{P_1}\cdot\frac{(N+P_1(1-\rho^{*2}))(P_1+P_2+2\rho^*\sqrt{P_1P_2}+N)}{(P_2+\rho^*\sqrt{P_1P_2})^2}\cdot\rho^{*2} \qquad (36)$$

then:

$$\frac{\gamma^2}{(1+\gamma^2)^2}=C(\rho^*). \qquad (37)$$

A solution for (37) is:

$$\gamma^2=\frac{1-2C(\rho^*)+\sqrt{1-4C(\rho^*)}}{2C(\rho^*)}. \qquad (38)$$

From (30) we now form

$$\rho=-\rho^*\sqrt{\frac{1-\sqrt{1-4C(\rho^*)}}{1-2C(\rho^*)+\sqrt{1-4C(\rho^*)}}}. \qquad (39)$$

It follows that those $\rho^*\geq 0$ are achievable for which both:

$$C(\rho^*)\leq 0.25 \qquad (40)$$

$$\rho\geq -1.0. \qquad (41)$$

For an achievable ρ^* it then follows by arguments as in section 1 that it is possible to set

$$R_2=\frac{1}{2}\ln\left(\frac{a_k}{a_{k+1}}\right)=\frac{1}{2}\ln\left(\frac{P_1+P_2+2\rho^*\sqrt{P_1P_2}+N}{N+P_1(1-\rho^{*2})}\right)$$

nats/transmission. (42)

Now in order to find the achievable rates R_1 we first note that:

$$I(\theta_1;Y^n|\theta_2)=H(Y^n|\theta_2)-H(Y^n|\theta_1,\theta_2) \qquad (43)$$

Here:

$$H(Y^n|\theta_1,\theta_2)=\sum_{k=1,n} H(Y_k|\theta_1,\theta_2,Y^{k-1})$$

$$=\sum_{k=1,n} H(Y_k|\theta_1,\theta_2,Y^{k-1},X_{1k},X_{2k})$$

$$=\sum_{k=1,n} H(Y_k|X_{1k},X_{2k})$$

$$=n.\frac{1}{2}\ln(2\pi eN) \text{ nats.} \quad (44)$$

And:

$$H(Y^n|\theta_2)=\sum_{k=1,n} H(Y_k|\theta_2,Y^{k-1})$$

$$=\sum_{k=1,n} H(Y_k|\theta_2,Y^{k-1},X_2^k)$$

$$=\sum_{k=1,n} H(Y_k-X_{2k}|\theta_2,(Y-X_2)^{k-1},X_2^k)$$

$$=\sum_{k=1,n} H(X_{1k}+N_k|\theta_2,(X_1+N)^{k-1})$$

$$=\sum_{k=1,n} H(X_{1k}+N_k|(X_1+N)^{k-1}). \quad (45)$$

Now define $U_k=X_{1k}+N_k$. We will now calculate the variance of the density $p(u_k|u^{k-1})$. Because

$$u_k=\sqrt{\frac{P_1}{1+\gamma^2}}\,(m_{1k}-\gamma m_{1k-1})+n_k \quad (46)$$

the covariance-matrix $C_u k$ equals:

$$C_u k=\begin{bmatrix} 1 & -a & 0 & 0 & \dots & 0 & 0 \\ -a & 1 & -a & 0 & \dots & 0 & 0 \\ 0 & -a & 1 & -a & \dots & 0 & 0 \\ 0 & 0 & -a & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -a \\ 0 & 0 & 0 & 0 & \dots & -a & 1 \end{bmatrix}.(P_1+N)$$

with

$$a=\frac{\gamma}{1+\gamma^2}\cdot\frac{P_1}{P_1+N} \quad (47)$$

If $D_u k=C_u^{-1} k$ then:

$$\text{var}\{p(u_k|u^{k-1})\}=\frac{1}{D_u k(k,k)}=\frac{\det(C_u k)}{\det(C_u k-1)}. \quad (48)$$

From (47) we find the recurrent relation:

$$\det(C_u k)=(P_1+N)\det(C_u k-1)-a^2(P_1+N)^2\det(C_u k-2). \quad (49)$$

For large k, $\text{var}\{p(u_k|u^{k-1})\}$ is independent of k. Therefore say:

$$\det C_u k=\text{constant}.q^k. \quad (50)$$

We find for (49) if we substitute (50) in it:

$$q^2-(P_1+N)q+a^2(P_1+N)^2=0. \quad (51)$$

The solution for this quadratic equation is:

$$q=(P_1+N)\left(\frac{1+\sqrt{1-4a^2}}{2}\right).$$

Therefore by (45), (48) and (49) for n large enough:

$$\frac{1}{n}H(Y^n|\theta_2)=\frac{1}{2}\ln\left(2\pi e(P_1+N)\left(\frac{1+\sqrt{1-4a^2}}{2}\right)\right). \quad (52)$$

Hence

$$\frac{1}{n}I(\theta_1;Y^n|\theta_2)=\frac{1}{2}\ln\left(\frac{P_1+N}{N}\cdot\frac{1+\sqrt{1-4C(\rho^*)\left(\frac{P_1}{P_1+N}\right)^2}}{2}\right). \quad (53)$$

Therefore a code with codewords x_1^n and arbitrary small probability of error exists with

$$R_1<\frac{1}{2}\ln\left(\frac{P_1+N}{N}\cdot\frac{1+\sqrt{1-4C(\rho^*)\left(\frac{P_1}{P_1+N}\right)^2}}{2}\right). \quad (54)$$

We now conclude:

For $\rho^*\geq 0$ such that

$C(\rho^*)\leq 0.25$ and

$\rho\geq -1.0$

(R_1,R_2) is achievable for a MAC with semi-feedback where

$$R_1=\frac{1}{2}\ln\left(\frac{P_1+N}{N}\cdot\frac{1+\sqrt{1-4C(\rho^*)\left(\frac{P_1}{P_1+N}\right)^2}}{2}\right) \text{ nats/transmission} \quad (55)$$

$$R_2=\frac{1}{2}\ln\left(\frac{P_1+P_2+2\rho^*\sqrt{P_1P_2}+N}{N+P_1(1-\rho^{*2})}\right) \text{ nats/transmission.} \quad (56)$$

The curve formed by these achievable rate-points is plotted in figure 3a for a MAC for which $P_1=P_2=10$ and $N=1$. It is the curve with parameter $\alpha=0$. This parameter indicates with what fraction of its power encoder 2 is not participating in the scheme we have just presented. With this power (αP_2) encoder 2 forms a random code of block-length n whose letters are drawn independently from zero-mean normal distribution with variance αP_2. The decoder now first treats this code letters as channel noise and first decodes the messages sent in the original scheme. After subtraction of the inputs caused by these messages the encoder sees only the codeword which was transmitted with power αP_2 and the added channel noise. This code can be decoded at rates up to:

$$\frac{1}{2}\ln\left(\frac{\alpha P_2+N}{N}\right) \text{ nats/transmission.} \quad (57)$$

Hence if we replace in (55) and (56) P_2 by $\bar{\alpha}P_2$ and N by

$N+\alpha P_2$ and add up (57) and (57) we find with $\bar{\alpha}=1-\alpha$ that:

(R_1,R_2) is achievable for a MAC with semi-feedback where:

$$R_1=\frac{1}{2}\ln\left(\frac{P_1+N+\alpha P_2}{N+\alpha P_2}\cdot\frac{1+\sqrt{1-4C'(\bar{\alpha},\rho^*)\left(\frac{P_1}{P_1+\alpha P_2+N}\right)^2}}{2}\right)$$

nats/transmission

$$R_2=\frac{1}{2}\ln\left(\frac{P_1+P_2+2\rho^*\sqrt{\bar{\alpha}P_1P_2}+N}{N+\alpha P_2+P_1(1-\rho^{*2})}\cdot\frac{\alpha P_2+N}{N}\right)\text{ nats/transmission}$$

for $0\le\alpha\le 1$ and $\rho^*\ge 0$ such that:

$$C'(\bar{\alpha},\rho^*)\le 0.25$$

and:

$$\rho=-\rho^*\sqrt{\frac{1+\sqrt{1-4C'(\bar{\alpha},\rho^*)}}{1-2C'(\bar{\alpha},\rho^*)+\sqrt{1-4C'(\bar{\alpha},\rho^*)}}}\ge -1.0$$

where:

$$C'(\bar{\alpha},\rho^*)=\frac{\bar{\alpha}P_2}{P_1}\cdot\frac{(N+\alpha P_2+P_1(1-\rho^{*2}))(P_1+P_2+2\rho^*\sqrt{\bar{\alpha}P_1P_2}+N)}{(\bar{\alpha}P_2+\rho^*\sqrt{\bar{\alpha}P_1P_2})^2}\cdot\rho^{*2}$$

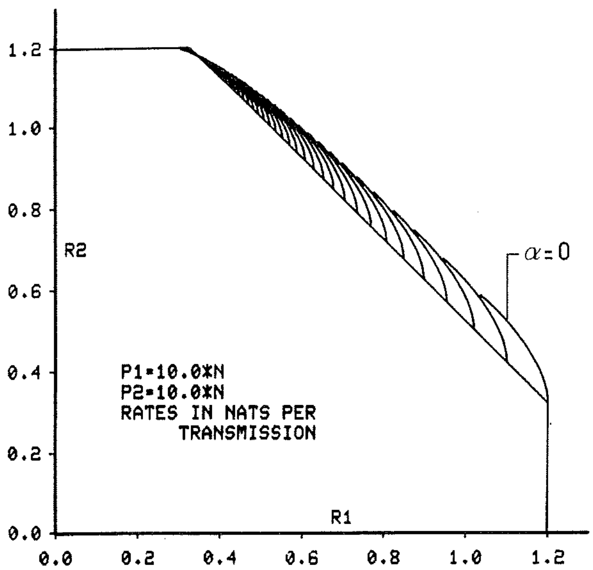

Fig.3a. Achievable rate region.

The curves formed by these achievable rate pairs are also plotted in figure 3a with α as a parameter. Similar curves for transmitter powers $P_1=10N$ and $P_2=5N$ can be found in figure 3b and for $P_1=5N$ and $P_2=10N$ in figure 3c.
It should be noted that for the AWGN MAC with semi-feedback also the Cover-Leung region [1] is achievable, as is proved by Willems and van der Meulen [6]. The coding scheme in the present manuscript however has the property that it is semi-constructive. This is not the case with the scheme in Willems and van der Meulen [6].

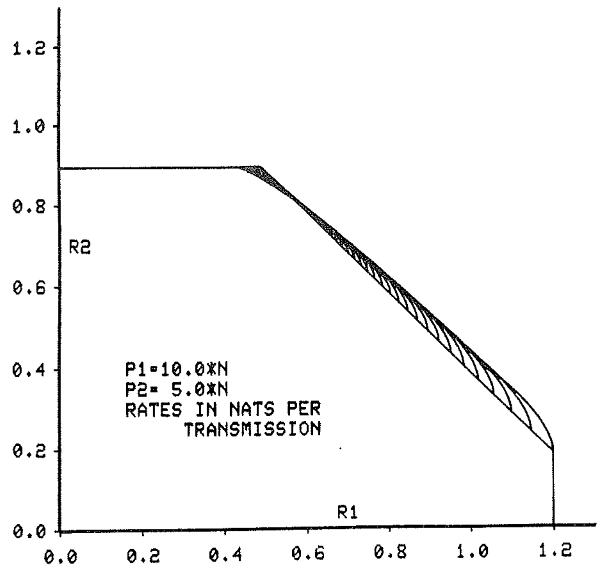

Fig.3b. Achievable rate region.

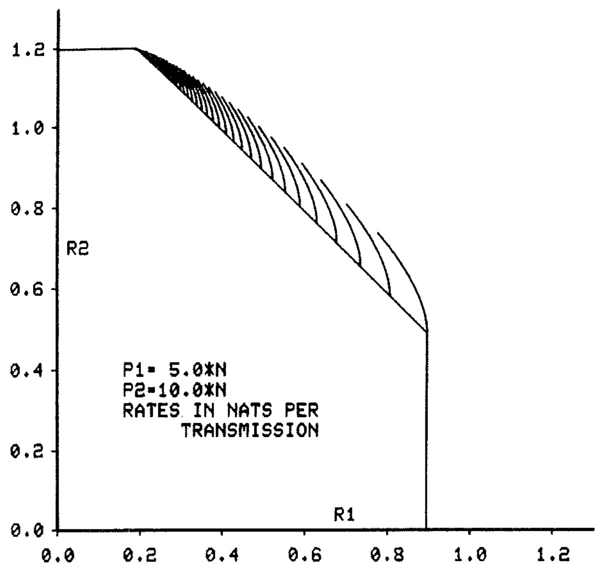

Fig.3c. Achievable rate region.

3. REFERENCES

[1] T.M.Cover and C.S.K.Leung, "An achievable rate region for the multiple-access channel with feedback", IEEE Trans.Inform.Theory, vol.IT-27, pp.292-298, May 1981.

[2] G.Dueck, "Semi-feedback for multi-user channels", manuscript, Universität Bielefeld, West Germany, April 1979.

[3] N.T.Gaarder and J.K.Wolf, "The capacity region of a multiple-access discrete memoryless channel can increase with feedback", IEEE Trans.Inform.Theory, vol.IT-21, pp.100-102, Jan.1975.

[4] L.H.Ozarow, "Coding and capacity for additive white Gaussian noise multi-user channels with feedback", Ph.D.dissertation Massachusetts Institute of Technology, May 1979.

[5] J.P.M.Schalkwijk, "A coding scheme for additive noise channels with feedback - part II: band-limited signals", IEEE Trans.Inform.Theory, vol.IT-12, pp.183-189, April 1966.

[6] F.M.J.Willems and E.C.van der Meulen, "Partial feedback for the discrete memoryless multiple access channel", IEEE Trans.Inform.Theory, vol.IT-29, pp.287-290, March 1983.

REMARK: This work was partially supported by Onderzoeksfonds K.U.Leuven, Project OT/VI/17.

THANKS to Mrs. Bijl-Wind for her help in preparing this manuscript.