# DEPENDENCE BALANCE BOUNDS

Frans M.J. Willems
(joint work w. Andries Hekstra (NXP))

Eindhoven University of Technology

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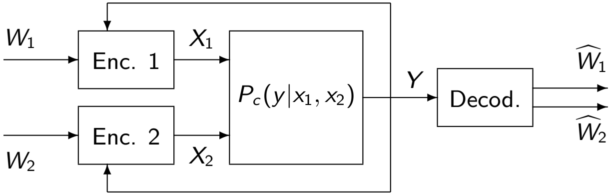
# Multiple-Access Channel with Feedback






Message index $W_1$ is uniform over $\{1, 2, \cdots, M_1\}$ and index $W_2$ is uniform over $\{1, 2, \cdots, M_2\}$.
Encoders:

$$\begin{aligned} X_{1n} &= E_{1n}(W_1, Y_1, Y_2, \cdots, Y_{n-1}), \\ X_{2n} &= E_{2n}(W_2, Y_1, Y_2, \cdots, Y_{n-1}), n = 1, 2, \cdots, N. \end{aligned}$$

Channel $\{\mathcal{X}_1 \times \mathcal{X}_2, P_c(y|x_1, x_2), \mathcal{Y}\}$ is discrete and memoryless.
Decoder:

$$(\widehat{W}_1, \widehat{W}_2) = D(Y_1, Y_2, \cdots, Y_N).$$

# Achievability, Capacity Region



Error probability:

$$P_e = \Pr\{(\widehat{W}_1, \widehat{W}_2) \neq (W_1, W_2)\}.$$

A rate pair $(R_1, R_2)$ with nonnegative $R_1$ and $R_2$ is **achievable** for a MAC with FB if for any $\epsilon > 0$ there exist for all $N$ large enough encoders and a decoder with

$$\begin{aligned} \log_2 M_1 &\geq N(R_1 - \epsilon), \\ \log_2 M_2 &\geq N(R_2 - \epsilon), \\ P_e &\leq \epsilon. \end{aligned}$$

The set of achievable rate pairs $(R_1, R_2)$ is the **capacity region** $\mathcal{R}_f$.

# An Inner and an Outer Bound for Capacity Region $\mathcal{R}_f$





**Theorem (Cover-Leung (1981) Inner Bound)**

$$\begin{aligned}\mathcal{R}_f \supseteq \{(R_1,R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1;Y|X_2,U),\\ & 0 \leq R_2 \leq I(X_2;Y|X_1,U),\\ & R_1+R_2 \leq I(X_1,X_2;Y),\\ & \textit{for } P(u,x_1,x_2,y) = P(u)P(x_1|u)P(x_2|u)P_c(y|x_1,x_2)\}\end{aligned}$$

**Theorem (Cut-Set Outer Bound)**

$$\begin{aligned}\mathcal{R}_f \subseteq \{(R_1,R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1;Y|X_2),\\ & 0 \leq R_2 \leq I(X_2;Y|X_1),\\ & R_1+R_2 \leq I(X_1,X_2;Y),\\ & \textit{for } P(x_1,x_2,y) = P(x_1,x_2)P_c(y|x_1,x_2)\}\end{aligned}$$

**QUESTION:** Can arbitrary joint distributions $\{P(x_1,x_2), x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}$ be realized?

# A Result for the Binary Adder MAC



| $X_1$ | $X_2$ | $Y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 2 |

Note that $X_1 \equiv Y - X_2$ for the binary adder MAC.

**Theorem (W. (1982))**

*For MACs for which there exist a mapping $\phi$ such that $X_1 \equiv \phi(Y, X_2)$ the Cover-Leung region is the feedback capacity region, hence*

$$\begin{aligned}\mathcal{R}_f = \{(R_1, R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1; Y|X_2, U) = H(X_1|U), \\ & 0 \leq R_2 \leq I(X_2; Y|X_1, U), \\ & R_1 + R_2 \leq I(X_1, X_2; Y), \\ & \textit{for } P(u, x_1, x_2, y) = P(u)P(x_1|u)P(x_2|u)P_c(y|x_1, x_2)\}\end{aligned}$$

**THEREFORE** the Cover-Leung region is the capacity region for the binary adder MAC.

# Gaarder & Wolf Scheme





| $X_1$ | $X_2$ | $Y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 2 |

- Assume that first the encoders independently transmit $k$ binary, uniformly distributed, digits over the adder channel.
- If a 0 or 2 is received, the decoder understands.
- However if the output of the channel is 1 (this happens with probability 1/2), the encoders have to resolve 1 bit of uncertainty extra to the decoder. They can however do this later in full cooperation, transmitting $\log_2 3$ bits/transmission.
- Rates (Gaarder & Wolf (1975)):

$$R_1 = R_2 = \frac{k}{k + \frac{k/2}{\log_2 3}} = 0.7602 \text{ bits/transm.}$$

The system can operate in an **independent** mode but also in a **full-cooperation, dependent** mode.

**QUESTION:** How much dependence can be created?

# Preliminaries

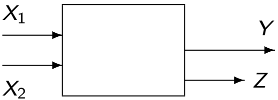
A. Some identities:

$$\begin{aligned} I(A;B|C) - I(A;B) &= H(B|C) - H(B|A,C) - H(B) + H(B|A) \\ &= I(B;C|A) - I(B;C) \\ &\quad \dots \\ &= I(C;A|B) - I(C;A). \end{aligned}$$

B: Let $Z$ be an extra output of the MAC. Hence

$$P(y,z|x_1,x_2) = P_c(y|x_1,x_2)P_e(z|x_1,x_2,y) \quad (1)$$

where $P_e$ is a channel with output $Z$ having $X_1$, $X_2$, and $Y$ as inputs.




**Definition**

A MAC with an extra output $Z$ is said to be in class $\mathcal{K}$ iff $I(X_1;X_2) = 0$ implies that $I(X_1;X_2|YZ) = 0$.

# Local Dependence Balance Bound

$$
\begin{aligned}
& I(W_1;W_2|(YZ)_n,(YZ)^{n-1}) - I(W_1;W_2|(YZ)^{n-1}) \\
&= I((YZ)_n;W_1|W_2,(YZ)^{n-1}) - I((YZ)_n;W_1|(YZ)^{n-1}) \\
&= H((YZ)_n|W_2,(YZ)^{n-1}) - H((YZ)_n|W_1,W_2,(YZ)^{n-1}) \\
&\quad -H((YZ)_n|(YZ)^{n-1}) + H((YZ)_n|W_1,(YZ)^{n-1}) \\
&\overset{(a)}{\le} H((YZ)_n|X_{2n},(YZ)^{n-1}) - H((YZ)_n|X_{1n},X_{2n},(YZ)^{n-1}) \\
&\quad -H((YZ)_n|(YZ)^{n-1}) + H((YZ)_n|X_{1n},(YZ)^{n-1}) \\
&= I((YZ)_n;X_{1n}|X_{2n},(YZ)^{n-1}) - I((YZ)_n;X_{1n}|(YZ)^{n-1}) \\
&= I(X_{1n};X_{2n}|(YZ)_n,(YZ)^{n-1}) - I(X_{1n};X_{2n}|(YZ)^{n-1}).
\end{aligned}
$$

In (a) we use that

$$
\begin{aligned}
H((YZ)_n|W_2,(YZ)^{n-1}) &= H((YZ)_n|W_2,X_{2n},(YZ)^{n-1}) \\
&\le H((YZ)_n|X_{2n},(YZ)^{n-1}),
\end{aligned}
$$

etc., and

$$
\begin{aligned}
H((YZ)_n|W_1,W_2,(YZ)^{n-1}) &= H((YZ)_n|W_1,W_2,X_{1n},X_{2n},(YZ)^{n-1}) \\
&= H((YZ)_n|X_{1n},X_{2n},(YZ)^{n-1}).
\end{aligned}
$$

## Lemma (Local dependence balance bound)

$$I(W_1; W_2|(YZ)_n, (YZ)^{n-1}) - I(W_1; W_2|(YZ)^{n-1}) \\ \leq \quad I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1}).$$

*Increase in dependence between $W_1$ and $W_2$ by observing $(YZ)_n$ is upper bounded by increase in dependence between $X_{1n}$ and $X_{2n}$ by observing $(YZ)_n$, all given $(YZ)^{n-1}$.*

## Lemma

*For a MAC with extra output $Z$ in class $K$ we have that $I(W_1; W_2|(YZ)^{n-1}) = 0$ for all $n = 1, 2, \cdots, N$.*

**PROOF:** By induction.

1. First note that $I(W_1; W_2) = 0$.
2. Let $I(W_1; W_2|(YZ)^{n-1}) = 0$ for some $n = 1, 2, \cdots, N$. Then also $I(X_{1n}; X_{2n}|(YZ)^{n-1}) = 0$ and for a MAC with extra output $Z$ in $\mathcal{K}$ this implies that $I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) = 0$. Now from the local DBB it follows that $I(W_1; W_2|(YZ)_n, (YZ)^{n-1}) = 0$.

# Consequences of the Local DBB

Using standard techniques we can now prove:

**Theorem**

*An outer bound for a MAC with FB is*

$$\begin{aligned}\mathcal{R}_f \subseteq \{(R_1, R_2) \quad : \quad & 0 \le R_1 \le I(X_1; YZ|X_2, U), \\ & 0 \le R_2 \le I(X_2; YZ|X_1, U), \\ & R_1 + R_2 \le I(X_1, X_2; Y), \text{ for} \\ & P(u, x_1, x_2, y, z) = P(u)P(x_1|u)P(x_2|u)P(y, z|x_1, x_2)\}\end{aligned}$$

*if the MAC with an extra output $Z$ is in class $\mathcal{K}$.*

**CONSEQUENCE:** A MAC for which there is a function $\phi$ such that $X_1 = \phi(Y, X_2)$ with extra output $Z = X_1$ is in class $\mathcal{K}$, since $I(X_1; X_2|Y, Z) = I(X_1; X_2|Y, X_1) = 0$.
By the above theorem again the Cover-Leung region is the capacity region for such channels.

# Global Dependence Balance Bound

$$\begin{aligned}
0 &\leq I(W_1; W_2|(YZ)^N) - I(W_1; W_2) \\
&= \sum_{n=1}^{N} [I(W_1; W_2|(YZ)^{n-1}), (YZ)_n) - I(W_1; W_2|(YZ)^{n-1})] \\
&\overset{(b)}{=} \sum_{n=1}^{N} [I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1})].
\end{aligned}$$

where (b) follows from the local DBB.

**Lemma (Global dependence balance bound)**

$$\sum_{n=1}^{N} [I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1})] \geq 0.$$

# A Consequence of the Global DBB

Using standard techniques we can now prove:

**Theorem**

*An outer bound for a MAC with FB is*

$$\begin{aligned}\mathcal{R}_f \subseteq \{(R_1, R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1; YZ|X_2, U), \\ & 0 \leq R_2 \leq I(X_2; YZ|X_1, U), \\ & R_1 + R_2 \leq I(X_1, X_2; Y), \ \textit{for} \\ & P(u, x_1, x_2, y, z) = P(u)P(x_1, x_2|u)P(y, z|x_1, x_2) \\ & \textit{such that } I(X_1; X_2|YZ, U) \geq I(X_1; X_2|U)\}\end{aligned}$$

*if the MAC has an extra output $Z$.*

**DEPENDENCE BALANCE:** $I(X_1; X_2|YZ, U) \geq I(X_1; X_2|U)$, i.e. produced dependence cannot be smaller than consumed dependence.

# Concluding Remarks



- Bounds and capacity regions for MACs with FB (Hekstra-W. (1985, 1989)).
- Similar results for Two-Way Channels, especially for the binary multiplying channel (Hekstra-W. (1985, 1989)).
- Gaussian MAC and Interference Channels with FB (Gastpar-Kramer (2006, 2007))
- Recent results on MACs with FB, MACs with noisy FB or with user cooperation, Interference Channels with user cooperation, Gaussian cases (Tandon-Ulukus (2007 ...)
- Relay Channel?
- Does something similar exist for the broadcast channel?