Wigner distribution and fractional Fourier transform for two-dimensional symmetric optical beams

Tatiana Alieva and Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands

Abstract

A useful relationship between the fractional Fourier transform power spectra of a two-dimensional symmetric optical beam on the one hand and its Wigner distribution on the other is established. This relationship allows a significant simplification of the standard procedure for the reconstruction of the Wigner distribution from the field intensity distributions in the fractional Fourier domains. The Wigner distribution of a symmetric optical beam is analyzed, both in the coherent and in the partially coherent case.

OCIS codes: 070.0070, 070.2590

1 Introduction

The characterization of coherent and partially coherent optical fields by means of the Wigner distribution [1, 2, 3] became a usual approach since the method of its reconstruction using phase-space tomography has been proposed [4, 5]. Nevertheless, the application of the Wigner distribution for the analysis of optical signals has some weak points. First of all, the Wigner distribution of a two-dimensional optical field is a four-dimensional function, which is difficult to represent graphically. Beside that, the reconstruction of this function using phase-space tomography demands to perform twice the complicated inverse Radon transform. These difficulties could be minimized if the optical field exhibits a certain symmetry, which is the case for a variety of optical beams used in practice.

In this paper we analyze the structure of the Wigner distribution of point-symmetric and rotationally symmetric optical beams and we propose a simple method for its reconstruction from the measurements of the beam intensity distributions at the different transversal planes during propagation through a first-order optical system. This method can be used if – due the physical nature of the optical field – we know that such symmetries occur. As an example of a first-order optical system we consider one that performs a two-dimensional fractional Fourier transformation. The generalization to other first-order systems described by the canonical integral transform can be made by rescaling the intensity distributions. In Section 2 we derive a useful relationship between the Wigner distribution and the fractional Fourier transform, which relationship significantly simplifies the reconstruction of the Wigner distribution in the cases of point-symmetric and rotationally symmetric beams, as will be shown in Sections 3 and 4. In

Section 5 we discuss the generalization of the results to the case of partially coherent symmetric optical fields.

2 Rotation of the Wigner distribution under two-dimensional fractional Fourier transformation

In general the Wigner distribution (WD) $W_f(\boldsymbol{r},\boldsymbol{q}) = W_f(x_1, x_2, u_1, u_2)$ of a two-dimensional function $f(\boldsymbol{r}) = f(x_1, x_2)$ – which in the context of this paper corresponds to the complex field amplitude of a coherent field – is defined as

$$W_f(\boldsymbol{r},\boldsymbol{q}) = \int_{-\infty}^{\infty} f(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}')f^*(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}')\exp(-i2\pi\boldsymbol{q}\cdot\boldsymbol{r}')d\boldsymbol{r}', \tag{1}$$

or, equivalently, expressed in terms of the Fourier transform (FT) $F_{\pi/2}(\boldsymbol{q}) = F_{\pi/2}(u_1, u_2)$ of $f(\boldsymbol{r})$, as

$$W_f(\boldsymbol{r},\boldsymbol{q}) = \int_{-\infty}^{\infty} F_{\pi/2}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{q}')F^*_{\pi/2}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{q}')\exp(i2\pi\boldsymbol{q}'\cdot\boldsymbol{r})d\boldsymbol{q}', \tag{2}$$

where

$$F_{\pi/2}(\boldsymbol{q}) = \int_{-\infty}^{\infty} f(\boldsymbol{r})\exp(-i2\pi\boldsymbol{q}\cdot\boldsymbol{r})d\boldsymbol{r}. \tag{3}$$

Note that if we put $\boldsymbol{q} = \boldsymbol{0}$ in Eq. (2), we have the following relationship between the WD $W_f(x_1, x_2, u_1, u_2)$ along the plane $u_1 = u_2 = 0$ and the two-dimensional FT $F_{\pi/2}(u_1, u_2)$:

$$\begin{aligned} W_f(\boldsymbol{r},\boldsymbol{0}) \;=\; W_f(x_1, x_2, 0, 0) \;=\; & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{\pi/2}(\tfrac{1}{2}u_1', \tfrac{1}{2}u_2')F^*_{\pi/2}(-\tfrac{1}{2}u_1', -\tfrac{1}{2}u_2') \\ & \times\ \exp[i2\pi(u_1'x_1 + u_2'x_2)]du_1'du_2'. \end{aligned} \tag{4}$$

Let us consider the WD under fractional Fourier transformation. The two-dimensional fractional FT $F_{\alpha_1,\alpha_2}(u_1, u_2)$ of a function $f(x_1, x_2)$ can be defined as [6, 7, 8]

$$\begin{aligned} F_{\alpha_1,\alpha_2}(u_1, u_2) &= R^{\alpha_1,\alpha_2}[f(x_1, x_2)](u_1, u_2) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\alpha_1, x_1, u_1)K(\alpha_2, x_2, u_2)f(x_1, x_2)dx_1dx_2, \end{aligned} \tag{5}$$

where the partial kernels $K(\alpha_n, x_n, u_n)$ $(n = 1, 2)$ are given by

$$K(\alpha_n, x_n, u_n) = \frac{\exp(i\alpha_n)/2}{\sqrt{i\sin\alpha_n}}\exp\left[i\pi\frac{(x_n^2 + u_n^2)\cos\alpha_n - 2u_nx_n}{\sin\alpha_n}\right]. \tag{6}$$

Note that, in particular, $F_{0,0}(u_1, u_2) = f(u_1, u_2)$, and that $F_{\pi/2,\pi/2}(u_1, u_2)$ corresponds to the normal two-dimensional FT.

It is well known (see, for example, [4] or [9]) that the fractional FT corresponds to a rotation of the WD in the space-frequency planes (x_1, u_1) and (x_2, u_2). This rotation can best be described by introducing polar coordinates in these planes:

$$\begin{array}{ll} x_1 = R_1 \cos\beta_1 & x_2 = R_2 \cos\beta_2 \\ u_1 = R_1 \sin\beta_1 & u_2 = R_2 \sin\beta_2. \end{array} \tag{7}$$

We will use the notation $\widetilde{W}_f(R_1, \beta_1, R_2, \beta_2)$ for the representation of the WD in this coordinate system. Since the two-dimensional fractional FT (5) corresponds to a rotation in the two space-frequency planes (x_1, u_1) and (x_2, u_2) through the angles α_1 and α_2, respectively,

$$\begin{array}{lll} f(x_1, x_2) & \longleftrightarrow & \widetilde{W}_f(R_1, \beta_1, R_2, \beta_2) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of WD} \\ R^{\alpha_1,\alpha_2}[f] & \longleftrightarrow & \widetilde{W}_{R^{\alpha_1,\alpha_2}[f]}(R_1, \beta_1, R_2, \beta_2) = \widetilde{W}_f(R_1, \beta_1 + \alpha_1, R_2, \beta_2 + \alpha_2), \end{array} \tag{8}$$

and choosing $\beta_1 = \beta_2 = 0$ in this scheme, we obtain that

$$\begin{aligned} \widetilde{W}_f(R_1, \alpha_1, R_2, \alpha_2) &= \widetilde{W}_{R^{\alpha_1,\alpha_2}[f]}(R_1, 0, R_2, 0) = \widetilde{W}_{F_{\alpha_1,\alpha_2}}(R_1, 0, R_2, 0) \\ &= W_{F_{\alpha_1,\alpha_2}}(x_1, x_2, 0, 0). \end{aligned} \tag{9}$$

Thus, on the analogy of Eq. (4), we obtain that

$$\begin{aligned} \widetilde{W}_f(R_1, \alpha_1, R_2, \alpha_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{\pi/2+\alpha_1,\pi/2+\alpha_2}(\tfrac{1}{2}u_1', \tfrac{1}{2}u_2') F^*_{\pi/2+\alpha_1,\pi/2+\alpha_2}(-\tfrac{1}{2}u_1', -\tfrac{1}{2}u_2') \\ &\times \exp[i2\pi(u_1' R_1 + u_2' R_2)] du_1' du_2'. \end{aligned} \tag{10}$$

This equation shows a simple relationship between the fractional FT and the WD of a given function through the FT. It may be useful for the calculation of the WD, but its application in optics is unlikely because of the required phase measurements. The general method for the reconstruction of the WD from the fractional FT spectra $|F_{\alpha_1,\alpha_2}(u_1, u_2)|^2$ is based on the inverse Radon transform [4, 5]. Nevertheless, Eq. (10) can be used for the reconstruction of the WD of symmetric optical fields.

3 Wigner distribution of point-symmetric beams

Let us consider a point-symmetric function $f(\boldsymbol{r})$,

$$f(\boldsymbol{r}) = \pm f(\boldsymbol{r}), \tag{11}$$

which, with the $+$ sign, is often used as a model for an optical beam. It is easy to see that the WD of such a function is also point-symmetric:

$$W_f(-\boldsymbol{r}, -\boldsymbol{q}) = \pm W_f(\boldsymbol{r}, \boldsymbol{q}). \tag{12}$$

From Eq. (6) it follows that $K(\alpha_n, x_n, -u_n) = K(\alpha_n, -x_n, u_n)$, and we conclude that the fractional FT of a point-symmetric function is point-symmetric, as well:

$$F_{\alpha_1,\alpha_2}(-u_1, -u_2) = \pm F_{\alpha_1,\alpha_2}(u_1, u_2). \tag{13}$$

This implies that for point-symmetric functions Eq. (10) reduces to

$$\begin{aligned}\widetilde{W}_f(R_1, \alpha_1, R_2, \alpha_2) \quad &= \quad \pm \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_{\pi/2+\alpha_1,\pi/2+\alpha_2}(\tfrac{1}{2}u_1', \tfrac{1}{2}u_2')|^2 \\ &\qquad \times \; \exp[i2\pi(u_1' R_1 + u_2' R_2)] du_1' du_2'. \end{aligned} \tag{14}$$

The WD of a point-symmetric function can then be found as the inverse FT of its scaled fractional FT spectra $|F_{\alpha_1,\alpha_2}(u_1, u_2)|^2$. In order to do this, one has to fix the angle α_1, say, and measure or calculate the fractional power spectra for $\alpha_2 \in [0, \pi]$, then change to a different angle α_1 and repeat the procedure, and so on. Because of the symmetry of the WD [see Eq. (12)], the regions of the angles reduce to $\alpha_n \in [0, \frac{1}{2}\pi]$. In general, if we take N points for every angle α_n $(n = 1, 2)$, we need N^2 fractional FT spectra in order to reconstruct the WD.

4 Wigner distribution of rotationally symmetric beams

In this section we show that in the case of rotationally symmetric optical beams, described by

$$f(\boldsymbol{r}) = f(|\boldsymbol{r}|) = f(\rho_x) \tag{15}$$

with $\rho_x^2 = |\boldsymbol{r}|^2 = x_1^2 + x_2^2$, the number of fractional FT spectra that we need for the reconstruction of the WD, can be reduced to N. Note that a rotationally symmetric function is a particular case of the point-symmetric one and that, with the $+$ sign, Eq. (14) is valid for it, as well. Moreover, with $\rho_u^2 = u_1^2 + u_2^2$, its two-dimensional FT $F_{\pi/2}(u_1, u_2)$ reduces to the Hankel transform (HT) $H_{\pi/2}(\rho_u)$:

$$F_{\pi/2}(u_1, u_2) = H_{\pi/2}(\rho_u) = \int_0^{\infty} f(\rho_x) J_0(2\pi\rho_u\rho_x)\rho_x d\rho_x \tag{16}$$

with J_0 the 0-order Bessel function of the first kind.

The WD of the rotationally symmetric function $f(x_1, x_2) = f(\rho_x)$, expressed in the standard polar coordinates $(\rho_x, \theta_x, \rho_u, \theta_u)$, can be written as

$$\begin{aligned}\widehat{W}_f(\rho_x, \theta_x, \rho_u, \theta_u) \quad &= \quad \int_0^{\infty} \rho_x' d\rho_x' \int_0^{2\pi} d\theta_x' f(r_x^{(+)}) f^*(r_x^{(-)}) \\ &\qquad \times \; \exp[-i2\pi\rho_u\rho_x' \cos(\theta_u - \theta_x')] \end{aligned} \tag{17}$$

$$\begin{aligned} &= \quad \int_0^{\infty} \rho_u' d\rho_u' \int_0^{2\pi} d\theta_u' F_{\pi/2}(r_u^{(+)}) F_{\pi/2}^*(r_u^{(-)}) \\ &\qquad \times \; \exp[i2\pi\rho_x\rho_u' \cos(\theta_x - \theta_u')], \end{aligned} \tag{18}$$

where

$$
\begin{array}{ll}
r_x^{(+)} = \left[\rho_x^2 + \frac{1}{4}\rho_x'^2 + \rho_x\rho_x'\cos(\theta_x - \theta_x')\right]^{1/2} & r_u^{(+)} = \left[\rho_u^2 + \frac{1}{4}\rho_u'^2 + \rho_u\rho_u'\cos(\theta_u - \theta_u')\right]^{1/2} \\
r_x^{(-)} = \left[\rho_x^2 + \frac{1}{4}\rho_x'^2 - \rho_x\rho_x'\cos(\theta_x - \theta_x')\right]^{1/2} & r_u^{(-)} = \left[\rho_u^2 + \frac{1}{4}\rho_u'^2 - \rho_u\rho_u'\cos(\theta_u - \theta_u')\right]^{1/2}
\end{array}
\tag{19}
$$

and

$$
\begin{array}{llll}
x_1 = \rho_x\cos\theta_x & x_1' = \rho_x'\cos\theta_x' & u_1 = \rho_u\cos\theta_u & u_1' = \rho_u'\cos\theta_u' \\
x_2 = \rho_x\sin\theta_x & x_2' = \rho_x'\sin\theta_x' & u_2 = \rho_u\sin\theta_u & u_2' = \rho_u'\sin\theta_u'.
\end{array}
\tag{20}
$$

The explicit relationships between the two coordinate systems $(R_1, \alpha_1, R_2, \alpha_2)$ and $(\rho_x, \theta_x, \rho_u\theta_u)$ read [cf. Eqs. (7) and (20)]

$$
\begin{array}{lllll}
\rho_x\cos\theta_x & = & x_1 & = & R_1\cos\alpha_1 \\
\rho_x\sin\theta_x & = & x_2 & = & R_2\cos\alpha_2 \\
\rho_u\cos\theta_u & = & u_1 & = & R_1\sin\alpha_1 \\
\rho_u\sin\theta_u & = & u_2 & = & R_2\sin\alpha_2.
\end{array}
\tag{21}
$$

It is now easy to show that

$$
\widehat{W}_f(\rho_x, \theta_x - \gamma, \rho_u, \theta_u) = \widehat{W}_f(\rho_x, \theta_x, \rho_u, \theta_u + \gamma), \tag{22}
$$

which implies that we can treat the four-dimensional WD as a three-dimensional function, because

$$
\widehat{W}_f(\rho_x, \theta_x - \theta_u, \rho_u, 0) = \widehat{W}_f(\rho_x, \theta_x, \rho_u, \theta_u) = \widehat{W}_f(\rho_x, 0, \rho_u, \theta_u - \theta_x). \tag{23}
$$

In particular we have the relationship

$$
\widehat{W}_f(\rho_x, \theta, \rho_u, \theta) = \widehat{W}_f(\rho_x, 0, \rho_u, 0). \tag{24}
$$

From Eq. (23) we conclude that, in order to find $\widehat{W}_f(\rho_x, \theta_x, \rho_u, \theta_u)$, we can put either $\theta_x = 0$ or $\theta_u = 0$; in [10], in which the WD of a circular aperture was studied, the choice $\theta_x = 0$ was made. In the present paper we take $\theta_u = 0$, which is the same as putting $\alpha_2 = 0$ if the coordinate system $(R_1, \alpha_1, R_2, \alpha_2)$ is used, see Eq. (21). Therefore, in order to reconstruct the WD completely, we have to find only $\widetilde{W}_f(R_1, \alpha_1, R_2, 0)$. From Eq. (14) we then get the FT relationship

$$
\widetilde{W}_f(R_1, \alpha_1, R_2, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_{\pi/2+\alpha_1, \pi/2}(\tfrac{1}{2}u_1', \tfrac{1}{2}u_2')|^2 \exp[i2\pi(u_1'R_1 + u_2'R_2)]du_1'du_2', \tag{25}
$$

which reduces to a Hankel transformation in the particular case that $\alpha_1 = 0$:

$$
\begin{array}{lll}
\widetilde{W}_f(R_1, 0, R_2, 0) & = & \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_{\pi/2, \pi/2}(\tfrac{1}{2}u_1', \tfrac{1}{2}u_2')|^2 \exp[i2\pi(u_1'R_1 + u_2'R_2)]du_1'du_2' \\
& = & \displaystyle\int_0^{\infty} |H_{\pi/2}(\tfrac{1}{2}\rho_u')|^2 J_0(2\pi R\rho_u')\rho_u' d\rho_u' \quad = \quad \widehat{W}_f(R, 0, 0, 0)
\end{array}
\tag{26}
$$

with $R^2 = R_1^2 + R_2^2$. From Eq. (25) we conclude that for the reconstruction of the WD of a rotationally symmetric function, we need only N fractional FT spectra $|F_{\alpha_1,0}(u_1,u_2)|^2$. Moreover, we note that, in general, the WD of rotationally symmetric functions cannot be obtained from measurements of the intensity distributions in rotationally symmetric first-order optical system.

If we put $\alpha_1 = \alpha_2 = \alpha$ in Eq. (14), we will find that the slices of the WD $\widehat{W}_f(\rho_x,\theta,\rho_u,\theta)$ for any $\theta_x = \theta_u = \theta$, depend only on $R = \sqrt{R_1^2 + R_2^2}$, and not on R_1 and R_2. Indeed, in the case of a rotationally symmetric function, the two-dimensional symmetric fractional FT (5), which is also rotationally symmetric, reduces to the fractional HT [11]

$$\begin{aligned} H_\alpha(\rho_u) &= R^\alpha[f(\rho_x)](\rho_u) \\ &= \frac{\exp(i\alpha)}{i\sin\alpha}\int_0^\infty \exp[i\pi(\rho_x^2+\rho_u^2)\cot\alpha]J_0\left(2\pi\frac{\rho_x\rho_u}{\sin\alpha}\right)f(\rho_x)\rho_x d\rho_x. \end{aligned} \tag{27}$$

For the special case $\alpha_1 = \alpha_2 = \alpha$, Eq. (14) then takes the form

$$\begin{aligned} \widetilde{W}_f(R_1,\alpha,R_2,\alpha) &= \int_{-\infty}^{\infty} |F_{\pi/2+\alpha,\pi/2+\alpha}(\tfrac{1}{2}u_1,\tfrac{1}{2}u_2)|^2 \exp[i2\pi(u_1R_1+u_2R_2)]du_1du_2 \\ &= \int_0^\infty |H_{\pi/2+\alpha}(\tfrac{1}{2}\rho_u)|^2 J_0(2\pi\rho_u R)\rho_u d\rho_u. \end{aligned} \tag{28}$$

Hence, a fractional Hankel transformation produces a rotation of the WD along the plane $\theta_x = \theta_u = \theta$, where the slice of the WD is rotationally symmetric:

$$\begin{aligned} \widehat{W}_f(\rho_x,\theta,\rho_u,\theta) &= \widehat{W}_f(R\cos\alpha,\theta,R\sin\alpha,\theta) \\ &= \int_0^\infty |H_{\pi/2+\alpha}(\tfrac{1}{2}\rho)|^2 J_0(2\pi\rho R)\rho d\rho \;=\; \widehat{W}_f(\rho_x,0,\rho_u,0) \end{aligned} \tag{29}$$

with

$$\begin{aligned} R^2 &= \rho_x^2 + \rho_u^2 \\ \tan\alpha &= \rho_u/\rho_x. \end{aligned} \tag{30}$$

Note that the fractional Hankel spectra $|H_\alpha(\rho_u)|^2$ do not add any new information about the WD to what we have already obtained from the Hankel spectrum [see Eq. (26)]. We also remark that, as it follows for example from Eq. (17), only the WD of the Gaussian beam $f(\rho_x) = A\exp(-a\rho_x^2)$ does not depend on the angles θ_x and θ_u and can thus be represented as a function of R.

5 Partially coherent symmetric beams

Although the previous considerations have been devoted to coherent symmetric fields, the main results are also valid for the case of partially coherent ones. In order to prove this, let us represent the two-point correlation function $G(\boldsymbol{r}_1, \boldsymbol{r}_2)$ of a partially coherent field as a linear superposition of the orthonormal modes $g_m(\boldsymbol{r})$ [12]:

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{m=0}^{\infty} \lambda_m g_m(\boldsymbol{r}_1) g_m^*(\boldsymbol{r}_2). \tag{31}$$

The WD in the case of the partially coherent fields [2] is given by

$$W(\boldsymbol{r}, \boldsymbol{q}) = \int_{-\infty}^{\infty} G(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') \exp(-i2\pi \boldsymbol{q} \cdot \boldsymbol{r}') d\boldsymbol{r}', \tag{32}$$

and can be expressed as

$$W(\boldsymbol{r}, \boldsymbol{q}) = \sum_{m=0}^{\infty} \lambda_m w_m(\boldsymbol{r}, \boldsymbol{q}), \tag{33}$$

where

$$w_m(\boldsymbol{r}, \boldsymbol{q}) = \int_{-\infty}^{\infty} g_m(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}') g_m^*(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') \exp(-i2\pi \boldsymbol{q} \cdot \boldsymbol{r}') d\boldsymbol{r}'. \tag{34}$$

It is easy to see that if all modes $g_m(\boldsymbol{r})$ exhibit a certain type of symmetry, then the WD of the partially coherent field has the same properties as the WD of a coherent field with the same symmetry. Because of the orthonormality of the modes, one can conclude that if $G(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is point-symmetric [cf. Eq. (11)],

$$G(-\boldsymbol{r}_1, -\boldsymbol{r}_2) = \pm G(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{35}$$

or rotationally symmetric [cf. Eq. (15)],

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = G(|\boldsymbol{r}_1|, |\boldsymbol{r}_2|), \tag{36}$$

then the modes $g_m(\boldsymbol{r})$ have the same type of symmetry, and vice versa. The results obtained for coherent symmetric beams in the previous sections are thus also valid in the case of partially coherent ones, if the symmetry of their correlation functions can be expressed by Eq. (35) or (36).

6 Conclusions

It has been shown that symmetry of the complex field amplitude or the correlation function of an optical field leads to a certain symmetry of its WD that significantly simplifies the task of its reconstruction from the two-dimensional fractional FT spectra.

In particular we have obtained that the WD for a field with point symmetry is related to the fractional FT spectra through a Fourier transformation. This implies that, unlike in the general case, one need not apply the complicated inverse Radon transform in this point-symmetric case. Moreover the point symmetry reduces the region of angles α_n where the fractional FT spectra have to be measured, from $[0, \pi]$ to $[0, \frac{1}{2}\pi]$. In the case of rotational symmetry the number of required fractional FT spectra reduces even more: if for a point-symmetric field one needs N^2

spectra, then in the case of rotational symmetry only N fractional asymmetric FT spectra are required in order to reconstruct the WD.

It has been shown that, in spite of the rotational symmetry of the optical field, its WD cannot be reconstructed from the fractional HT alone, but that the asymmetric fractional FT spectra are needed. In particular this means, as was also discussed in [13], that rotationally symmetric fields cannot be completely identified by their propagation in the paraxial approximation in free space, if only the information about the intensity is used.

The authors can be reached at the address on the title page, by telephone at +31 40 2473319, by fax at +31 40 2448375, or by e-mail at M.J.Bastiaans@ele.tue.nl.

References

[1] E. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749-759 (1932).

[2] M. J. Bastiaans, "The Wigner distribution function applied to optical signals and systems," Opt. Commun. **25**, 26-30 (1978).

[3] W. Mecklenbräuker and F. Hlawatsch, Eds., *The Wigner Distribution: Theory and Applications in Signal Processing* (Elsevier, Amsterdam, 1997).

[4] M. G. Raymer, M. Beck, and D. F. McAlister, "Complex wave-field reconstruction using phase-space tomography," Phys. Rev. Lett. **72**, 1137-1140 (1994).

[5] D. F. McAlister, M. Beck, L. Clarke, A. Mayer, and M. G. Raymer "Optical phase retrieval by phase-space tomography and fractional-order Fourier transforms," Opt. Lett. **20**, 1181-1183 (1995).

[6] Y. B. Karasik, "Expression of the kernel of a fractional Fourier transform in elementary functions," Opt. Lett. **19**, 769-770 (1994).

[7] A. Sahin, H. M. Ozaktas, and D. Mendlovic, "Optical implementations of two-dimensional fractional Fourier transforms and linear canonical transforms with arbitrary parameters," Appl. Opt. **37**, 2130-2141 (1998).

[8] A. Sahin, M. A. Kutay, and H. M. Ozaktas, "Nonseparable two-dimensional fractional Fourier transform," Appl. Opt. **37**, 5444-5453 (1998).

[9] A. W. Lohmann, "Image rotation, Wigner rotation and the fractional Fourier transform," J. Opt. Soc. Am. A **10**, 2181-2186 (1993).

[10] M. J. Bastiaans and P. G. J. van de Mortel, "Wigner distribution function of a circular aperture," J. Opt. Soc. Am. A **13**, 1698-1703 (1996).

[11] L. Yu, Y. Lu, X. Zeng, M. Huang, M. Chen, W. Huang, and Z. Zhu, "Deriving the integral representation of a fractional Hankel transform from a fractional Fourier transform," Opt. Lett. **23**, 1158-1160 (1998).

[12] E. Wolf, "New theory of partial coherence in the space-frequency domain. I. Spectra and cross spectra of steady state sources," J. Opt. Soc. Am. **72**, 343-351 (1982).

[13] F. Gori, M. Santarsiero, and G. Guattari, "Coherence and the spatial distribution of intensity," J. Opt. Soc. Am. A **10**, 673-679 (1993).