Mode analysis in optics through fractional transforms

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Abstract

The relationship between the mode content and the fractional Fourier and fractional Hankel transforms of a function is established. It is shown that the Laguerre-Gauss spectrum of a rotationally symmetric wavefront can be determined from its fractional Hankel transforms taken at the optical axis.

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The Fourier transform, which gives information about the harmonic content of a wavefront, is one of the most important tools in wave optics. In certain applications however, for example in the analysis of laser cavity modes, knowledge of the mode content is even more important. Recently, different kinds of fractional integral transforms, closely related to the Fourier transform, were introduced in optical signal processing [1, 2]. They describe in the paraxial approximation of the scalar diffraction theory, the evolution of the complex field amplitude during wave propagation through quadratic refractive-index media such as lenses, fibers, mirrors, etc. Here we consider first-order optical systems performing a fractional Fourier transform (FT) or a fractional Hankel transform (HT) of the input complex field amplitude $f(\mathbf{x})$. A typical optical setup [3] contains a thin lens with focal length f, situated at the same distance $d = 2f\sin^2(\alpha/2)$ from the input and the output plane transversal to the optical axis. Applying cylindrical lenses with focal lengths f_1 and f_2 [4], one gets at the output plane the fractional FT at the angle $\alpha = (\alpha_1, \alpha_2)$ of the input complex field amplitude. In the case of rotational symmetry of the input wavefront, $f(\mathbf{x}) = f(x)$, a spherical lens performs a fractional Hankel transform at the angle α of the input complex field amplitude. The existence of inverse transforms allows us to determine the input wavefront from knowledge of the complex field amplitude at any plane transversal to the optical axis of the fractional FT or HT systems. The input complex field amplitude can also be determined from its Radon-Wigner transform [5], which corresponds to the intensity distribution at transversal planes of the fractional FT and fractional HT systems, taken in the region $\alpha \in [0, \pi]$ and $\alpha \in \left[0, \pi/2\right]$, respectively. In this case the parameter α as well as the coordinate $\mathbf{u}$ of the transform are considered as variables.

Here we consider the problem of reconstructing an optical signal from its fractional FT or HT, where the coordinate $\mathbf{u}$ at the plane transversal to the propagation axis is fixed and the angle α is considered as a variable. This problem corresponds to the input complex amplitude characterization from knowledge of the complex field amplitude at a line parallel to the optical axis of the corresponding system. We show how the information about the mode content of the input optical signal can be recovered from these measurements.

First we propose a method for the determination of the Hermite-Gauss mode spectra from the fractional Fourier transform of the investigated field for all possible combinations of the parameters $\alpha_1, \alpha_2 \in [0, 2\pi[$.

The two-dimensional fractional Fourier transform $F(u_1,u_2,\alpha_1,\alpha_2)$ at angles α_1 and α_2 of a function $f(\mathbf{x})$ is defined as

$$F(u_1,u_2,\alpha_1,\alpha_2) = R^{\alpha_1,\alpha_2}\left[f(\mathbf{x})\right](\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x})K_{\alpha_1}(x_1,u_1)K_{\alpha_2}(x_2,u_2)d\mathbf{x}, \qquad (1)$$

where

$$K_{\alpha_j}(x_j,u_j) = \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\ \exp\left(i\frac{\cos\alpha_j(x_j^2+u_j^2)-2x_ju_j}{2\sin\alpha_j}\right). \qquad (2)$$

We will suppose that the fractional FT operator acts independently on the coordinates x_1 and x_2. Thus the transversal plane amplitude $F(u_1,u_2,\alpha_1,\alpha_2)$ at a distance $2d$, where $d = 2f_1\sin^2(\alpha_1/2) = 2f_2\sin^2(\alpha_2/2)$ and f_1 and f_2 are the focal lengths of the cylindrical lenses situated at the distance d from the input plane, is the fractional FT of the input complex field amplitude $f(\mathbf{x})$ (for which $\alpha_1=\alpha_2=0$). We call such an optical system a fractional FT system and we suppose that α_1,α_2 change from 0 to 2π.

The eigenfunctions of the fractional FT at any angle α are the Hermite-Gauss modes $\Psi_n(x)$ defined by

$$\Psi_n(x) = \left(\sqrt{\pi}2^n n!\right)^{-1/2}\exp(-x^2/2)H_n(x), \qquad (3)$$

where $H_n(x)$ are the Hermite polynomials. Thus

$$R^{\alpha}\left[\Psi_n(x)\right](u) = \exp(-i\alpha n)\Psi_n(u), \qquad (4)$$

where $\exp(-i\alpha n)$ is the eigenvalue of the eigenfunction Ψ_n.

Since the Hermite-Gauss functions form a complete orthonormal set,

$$\int_{-\infty}^{\infty}\Psi_n(x)\Psi_m(x)dx = \begin{cases} 0 & n\neq m \\ 1 & n=m \end{cases}, \qquad (5)$$

a complex field amplitude $f(\mathbf{x})$ can be expanded as

$$f(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\Psi_n(x_1)\Psi_m(x_2), \qquad (6)$$

where

$$f_{n,m} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_n(x_1)\Psi_m(x_2)f(x_1,x_2)dx_1dx_2. \qquad (7)$$

Thus the set of coefficients $\left\{f_{n,m}\right\}$, which we will call the Hermite-Gauss spectrum, completely determines the function $f(\mathbf{x})$.

Using Eqs. (1), (4) and (6), we can write the expression for the fractional FT at angles α_1,α_2 of the input wavefront $f(\mathbf{x})$ in the form

$$\begin{aligned} F(u_1,u_2,\alpha_1,\alpha_2) &= R^{\alpha_1,\alpha_2}\left[f(\mathbf{x})\right](\mathbf{u}) \qquad (8)\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\exp(-i(\alpha_1 n+\alpha_2 m))\Psi_n(u_1)\Psi_m(u_2). \end{aligned}$$

Considering $F(u_1,u_2,\alpha_1,\alpha_2)$ as a function of α_j, it is easy to see that the set of products of the Hermite-Gauss coefficients $f_{n,m}$, and $\Psi_n(u_1)$ and $\Psi_m(u_2)$ for fixed coordinates u_1,u_2, composes the harmonic mode content of the periodic function $F(u_1,u_2,\alpha_1,\alpha_2)$. The inverse two-dimensional FT of $F(u_1,u_2,\alpha_1,\alpha_2)$ with respect to the parameters $\alpha_1$and α_2,

$$Z(\mathbf{u},\mathbf{p}) = \int_0^{2\pi}\int_0^{2\pi} R^{\alpha_1,\alpha_2}\left[f(\mathbf{x})\right](\mathbf{u})\exp(i(\alpha_1p_1+\alpha_2p_2))d\alpha_1d\alpha_2, \qquad (9)$$

produces the discrete-$\mathbf{p}$ function $Z(\mathbf{u},\mathbf{p})$,

$$Z(\mathbf{u},\mathbf{p}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\Psi_n(u_1)\Psi_m(u_2)\delta(p_1-n)\delta(p_2-m), \tag{10}$$

which for a fixed value $\mathbf{u} = \mathbf{u}_0$ corresponds to the Hermite-Gauss spectrum $f_{n,m}$ modulated by $\Psi_n(u_{1,0})\Psi_m(u_{2,0})$. This implies that $Z(\mathbf{u},\mathbf{p})$ can be different from 0 only at the points $p_1 = n$ and $p_2 = m$:

$$Z(\mathbf{u}_0, n, m) = f_{n,m}\Psi_n(u_{1,0})\Psi_m(u_{2,0}). \tag{11}$$

We conclude that the complex field amplitude at any transversal plane can be determined from knowledge of its values along the lines parallel to the optical axis of the fractional FT system.

Note that for a given value $\mathbf{u}_0$ the Hermite-Gauss function of certain indices n_j and m_j could be equal to 0, in which case one has to choose a different value $\mathbf{u}$ for recovering the Hermite-Gauss spectrum component f_{n_j,m_j}. For $\mathbf{u}_0 = 0$, for instance, all odd-order coefficients in Eq. (10) are zero, since $\Psi_{2n+1}(0) = 0$ for $n = 0, 1, \ldots$, while the even-order coefficients read

$$Z(0, 2n, 2m) = f_{2n,2m}\Psi_{2n}(0)\Psi_{2m}(0). \tag{12}$$

In that case we can only determine the spectrum of even-order Hermite-Gauss modes, which is sufficient only for an even function $f(\mathbf{x}) = f(-\mathbf{x})$. Note that the modulating coefficients are given by

$$\Psi_{2n}(0) = \frac{(-1)^n\sqrt{(2n)!}}{2^n\pi^{1/4}n!n}. \tag{13}$$

For the determination of all spectral components we then have to choose a different line $u = u_0 \neq 0$ parallel to the optical axis of the fractional FT system.

The simplest results can be obtained in the case of rotationally symmetric complex field amplitudes $f(\mathbf{x}) = f(x)$, where $x = \sqrt{x_1^2 + x_2^2}$. The two-dimensional fractional FT at identical angles for the two orthogonal coordinates of the rotationally symmetric function, reduces to the fractional Hankel transform of zero order.

The fractional Hankel transform of zero order at an angle α of a function $f(x)$, is defined as [2]

$$F(u,\alpha) = R^{\alpha}\left[f(x)\right](u) = \int_0^{\infty} f(x)H_{\alpha}(x,u)x\,dx, \tag{14}$$

where

$$H_{\alpha}(x,u) = (1 - i\cot\alpha)J_0(xu/\sin\alpha)\exp\left(i\cot\alpha(x^2+u^2)/2\right), \tag{15}$$

and $J_0(x)$ is a Bessel function of order 0. It is easy to see from Eq. (14) that $F(u,\alpha)$ is periodic in the parameter α with period π.

The eigenfunctions of the fractional HT at any angle α are the Laguerre-Gauss modes defined by

$$\Phi_n(x) = \sqrt{2}L_n(x^2)\exp(-x^2/2), \tag{16}$$

where $L_n(x)$ are the Laguerre polynomials. Thus

$$R^{\alpha}\left[\Phi_n(x)\right](u) = \exp(-2i\alpha n)\Phi_n(u), \tag{17}$$

where $\exp(-2i\alpha n)$ is the eigenvalue of the eigenfunction Φ_n.

Since the Laguerre-Gauss functions Φ_n – like the Hermite-Gauss functions Ψ_n, see Eq. (5) – form a complete orthonormal set,

$$\int_0^{\infty}\Phi_n(x)\Phi_m(x)x\,dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}, \tag{18}$$

a rotationally symmetric complex field amplitude $f(x)$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty} f_n \Phi_n(x), \tag{19}$$

where

$$f_n = \int_0^{\infty} \Phi_n(x)\, f(x) x dx. \tag{20}$$

Thus the set of coefficients $\{f_n\}$, which we will call the Laguerre-Gauss mode spectrum, completely determines the function $f(x)$.

Using Eqs. (14), (17) and (19) we can write the expression for the fractional HT at angle α of the input wavefront in the form

$$F(u,\alpha) = \sum_{n=0}^{\infty} f_n \exp(-2i\alpha n)\Phi_n(u).$$

Considering $F(u,\alpha)$ as a function of α, it is easy to see that the set of products of the Laguerre-Gauss coefficients f_n and $\Phi_n(u)$ for fixed u, composes the harmonic mode content of the periodic function $F(u,\alpha)$. The inverse FT of $F(u,\alpha)$ with respect to the parameter α,

$$S(u,\, p) = \int_0^{\pi} F(u,\alpha) \exp(i2\alpha p) d\alpha, \tag{21}$$

produces the discrete-p function $S(u,\, p)$,

$$S(u,\, p) = \sum_{n=0}^{\infty} f_n \Phi_n(u) \delta(p-n), \tag{22}$$

which for a fixed value $u = u_0$ corresponds to the Laguerre-Gauss spectrum f_n modulated by $\Phi_n(u_0)$. This implies that $S(u,\, p)$ can be different from 0 only at the points $p = n$:

$$S(u_0, n) = f_n \Phi_n(u_0). \tag{23}$$

Since $\Phi_n(0) = 1$, $S(0,n)$ reduces to the Laguerre-Gauss spectrum

$$S(0,n) = f_n. \tag{24}$$

This corresponds to knowledge of the complex field amplitude in the fractional HT system on the optical axis $F(0,\alpha)$ and applying an inverse FT.

So we conclude that a rotationally symmetric input field can be completely determined from knowledge of the complex field amplitude along the optical axis in the fractional HT system for $\alpha \in [0,\pi[$.

If only the average of $F(u,\alpha)$ on the coordinate u can be measured,

$$\langle F(u,\alpha) \rangle = \int_0^{\rho} F(u,\alpha) du \tag{25}$$

and $\langle F(u,\alpha) \rangle \neq 0$, then the Laguerre-Gauss spectrum can be determined as

$$f_n = \langle S(u,n) \rangle \,/\, \langle F(u,\alpha) \rangle\,. \tag{26}$$

Finally we conclude that the fractional FT and fractional HT treated as functions of the angle α, with the coordinate considered as a fixed parameter, allow us to recover the input function and determine its mode content.

References

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