Mapping–based width measures and uncertainty relations for periodic functions

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Abstract

Measures for the width of a periodic function are discussed, based on five different mappings of a (periodic) function on a circle to a (non–periodic) function on a line or line segment. The uncertainty relations corresponding to these measures are also obtained by using a generalization of the Cauchy–Schwarz inequality.

Key words: width measures, periodic functions, uncertainty relations, minimum uncertainty states

1 Introduction

It is well known that the witdh (measured in terms of the central second–order moment) of a function and that of its Fourier transform cannot be simultaneously arbitrarily small, as their product has to be equal to or greater than a given constant, according to Heisenberg's uncertainty relation [4,6,9]. Several attempts have been made at finding a suitable analogue of such relation between a periodic function and its Fourier coefficients, both in signal analysis and in quantum optics [18] within the context of photon number–phase relationships. The first task in solving such a problem is to define an appropriate measure for the width of a periodic function, and several

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options have been proposed [3]. One common option is the usual central second–order moment of the function restricted to a window of the size of a period. There is ambiguity, however, in the choice of this window. One way to resolve this ambiguity is to choose the window limits such that the width measure is minimized. While uncertainty relations using this measure have been found [5,13,17], they are not entirely satisfactory as they require information of the function other than the width measures.

An alternative and particularly intuitive measure for the width of periodic functions, which does not require the choice of a window, was first proposed by Bandilla and Paul [2], for quantum optics applications. This measure was later shown to satisfy a simple uncertainty relation [14]. Both the measure and the uncertainty relation were later interpreted in mechanical terms [15] (related to the centroid of a ring whose linear mass density is the squared modulus of the function) and further generalized [10,11]. It turns out, however, that this simple uncertainty relation is unattainable except in trivial limits, and that the true lower bounds are not given by a simple algebraic expression, but found in terms of nontrivial minimum uncertainty states [12,16]. The objectives of this paper are: (i) to study these measures as well as others in a unified framework, based on the mapping of a function on a circle (i.e. a periodic function) to a function on a line or line segment, and (ii) to derive associated uncertainty relations. Five particular mappings are considered. One advantage of this approach is that it is straightforward to define higher–order moments. Also, for some of the measures for the width discussed here, we find uncertainty relations that are more satisfactory than those known previously.

The layout of this paper is the following: The general procedure of using a mapping to define the width of a periodic function is described in Section 2. In Section 3, several mappings from the circle to the line are presented and discussed. Section 4 gives a generalization of the Cauchy–Schwarz inequality that serves as the basis for the rest of the manuscript. The corresponding uncertainty relations and, when available, minimum uncertainty states are found in Section 5 for periodic functions. Finally, some concluding remarks are given in Section 6.

2 The width of a periodic function

To characterize the width of a periodic function $f(\theta)$, we face two problems: (i) we have to find a proper definition of the width, and (ii) it is not easy to centralize second– and higher–order moments (as will be discussed at the end of this section). For a non–periodic function $f(t)$, things are clear: (i) the width d_t can be defined as the normalized (central) second–order moment,

$$d_t^2 = \frac{\displaystyle\int_{-\infty}^{+\infty} (t-t_\circ)^2 |f(t)|^2 \, \mathrm{d}t}{\displaystyle\int_{-\infty}^{+\infty} |f(t)|^2 \, \mathrm{d}t}, \tag{1}$$

and (ii) it is minimized if its derivative with respect to $t_\circ$ vanishes,

$$\int_{-\infty}^{+\infty} (t-t_\circ) |f(t)|^2 \, \mathrm{d}t = 0, \tag{2}$$

i.e. when the (central) first–order moment vanishes. From the latter relation the optimum value $t_\circ$ can easily be derived as the normalized first–order moment:

$$t_\circ = \frac{\displaystyle\int_{-\infty}^{+\infty} t |f(t)|^2 \, \mathrm{d}t}{\displaystyle\int_{-\infty}^{+\infty} |f(t)|^2 \, \mathrm{d}t}. \tag{3}$$

In the case of a periodic function $f(\theta)$, $f(\theta) = f(\theta + 2\pi)$, or – more easily – its centered version $f_\circ(\theta) = f(\theta + \theta_\circ)$, we will define the (positive) width parameter d_θ through

$$\xi^2(d_\theta) = \frac{\displaystyle\int_{-\pi+\theta_\circ}^{+\pi+\theta_\circ} \xi^2(\theta-\theta_\circ) |f(\theta)|^2 \, \mathrm{d}\theta}{\displaystyle\int_{2\pi} |f(\theta)|^2 \, \mathrm{d}\theta} = \frac{\displaystyle\int_{-\pi}^{+\pi} \xi^2(\theta) |f_\circ(\theta)|^2 \, \mathrm{d}\theta}{\displaystyle\int_{2\pi} |f_\circ(\theta)|^2 \, \mathrm{d}\theta}, \tag{4}$$

cf. Eq. (1), where $\xi(\theta)$ is an appropriately chosen mapping from the circle to the real axis. Of course, if the mapping $\xi(\theta)$ were periodic with period 2π, the exact integration bounds would be irrelevant, and we could simply write an integral over one arbitrary period 2π, like we did in the denominator of Eq. (4). We will assume that the mapping $\xi(\theta)$ is indeed periodic, $\xi(\theta) = \xi(\theta + 2\pi)$, and, moreover, that it is an odd function of θ, $\xi(\theta) = -\xi(-\theta)$.

With respect to $\theta_\circ$, the value of $\xi^2(d_\theta)$ is minimized for

$$\int_{2\pi} \xi(\theta-\theta_\circ)\xi'(\theta-\theta_\circ) |f(\theta)|^2 \, \mathrm{d}\theta = \int_{2\pi} \xi(\theta)\xi'(\theta) |f_\circ(\theta)|^2 \, \mathrm{d}\theta = 0, \tag{5}$$

cf. Eq. (2), with the requirement that

$$\int_{2\pi} [\xi(\theta-\theta_\circ)\xi'(\theta-\theta_\circ)]' |f(\theta)|^2 \, \mathrm{d}\theta = \int_{2\pi} [\xi'^2(\theta) + \xi(\theta)\xi''(\theta)] |f_\circ(\theta)|^2 \, \mathrm{d}\theta > 0. \tag{6}$$

From these expressions, the optimum value of $\theta_\circ$ can be found. Note that in general there is no direct relationship between the first–order moment of the function $f(\theta)$ and the optimum value of $\theta_\circ$ as defined above, leading to the minimum value of the width d_θ, like we had in the non–periodic case, cf. Eq. (3).

Although we will restrict ourselves in this paper to second–order moments, we remark that we can as well define normalized (central) higher–order moments by taking ξ^n with $n > 2$, in the numerator of Eq. (4).

3 Five possible mappings

In this section we consider the properties of five different mappings $\xi(\theta)$:

(1) stereographic projection: $\xi(\theta) = 2\tan(\theta/2)$
(2) cord distance: $\xi(\theta) = 2\sin(\theta/2)$
(3) vertical projection: $\xi(\theta) = \sin(\theta)$
(4) perimeter distance: $\xi(\theta) = \theta$
(5) central projection: $\xi(\theta) = \tan(\theta)$

Note that the mappings above are defined for the period $|\theta| < \pi$. In Fig. 1 we have illustrated these five different mappings. Note that for $|\theta| \leq \pi/2$, all mappings are monotonic and we have $|\sin(\theta)| \leq |2\sin(\theta/2)| \leq |\theta| \leq |2\tan(\theta/2)| \leq |\tan(\theta)|$. If $|\theta|$ passes the value $\pi/2$, the vertical projection and the central projection become less attractive; therefore, they should only be used for functions that are restricted to the interval $|\theta| \leq \pi/2$.

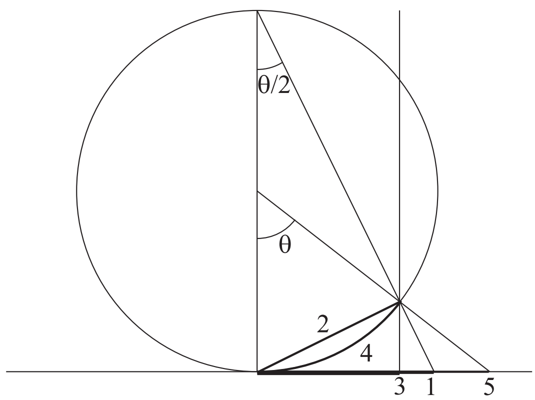

Fig. 1. Five possible mappings: 1. stereographic projection; 2. cord distance; 3. vertical projection; 4. perimeter distance; and 5. central projection.

3.1 Stereographic projection

Our first choice of $\xi(\theta)$ is the mapping

$$\xi(\theta) = 2\tan(\theta/2), \tag{7}$$

which maps the full circle $-\pi \leq \theta \leq \pi$ onto the complete axis $-\infty \leq \xi \leq \infty$. We may therefore profit from the knowledge of moments in the case of non–periodic functions, which are also defined on a complete axis. This mapping is analogous to the one that arises in discrete–time signal processing, where the z plane is mapped to the s plane by the bilinear mapping $sD = (z-1)/(z+1)$, with D an arbitrary sampling period, resulting in the relation $\omega D = 2\tan(\theta/2)$, which maps the unit circle $\exp(j\theta)$ in the z plane to the imaginary $j\omega$ axis in the s plane. As an example, we throughout consider in this section the rectangular function $f(\theta) = \mathrm{rect}(\theta/2a\pi)$, $0 < a < 1$, which equals 1 for $|\theta| < a\pi$ and 0 elsewhere. A straightforward calculation, starting from Eq. (4), leads to

$$2\tan(d_\theta/2) = 2\sqrt{\tan(a\pi/2)/(a\pi/2) - 1}. \tag{8}$$

Note that for small a we have $d_\theta \simeq a\pi/\sqrt{3}$, as can be expected, while for $a \to 1$ we have $d_\theta \to \pi$. Unfortunately, this projection does not allow an easy determination of the optimum value of $\theta_\circ$, except when $|f(\theta)| = |f(-\theta)|$.

3.2 Cord distance

Our second choice is the mapping

$$\xi(\theta) = 2\sin(\theta/2). \tag{9}$$

We again consider the example of a rectangular function $f(\theta) = \mathrm{rect}(\theta/2a\pi)$. The width is now

$$2\sin(d_\theta/2) = \sqrt{2[1 - \sin(a\pi)/a\pi]}. \tag{10}$$

Note that for small a we have $d_\theta \simeq a\pi/\sqrt{3}$ again, while for $a \to 1$ we have $d_\theta \to \pi/2$.

From Eqs. (5) and (6) we conclude that for this particular mapping, stationary values for d_θ occur if

$$2\int_{2\pi} \sin(\theta)|f(\theta+\theta_\circ)|^2\,\mathrm{d}\theta = 0, \tag{11}$$

$$\int_{2\pi} \cos(\theta)|f(\theta+\theta_\circ)|^2\,\mathrm{d}\theta > 0, \tag{12}$$

and the optimum value of $\theta_\circ$ can easily be found by evaluating the expression

$$\int_{2\pi} \mathrm{e}^{j\theta}\,|f(\theta)|^2\,\mathrm{d}\theta = \int_{2\pi} \mathrm{e}^{j(\theta+\theta_\circ)}\,|f(\theta+\theta_\circ)|^2\,\mathrm{d}\theta$$

and requiring that $\theta_\circ$ is chosen such that

$$\mathrm{Im}\left\{\mathrm{e}^{-j\theta_\circ}\int_{2\pi}\mathrm{e}^{j\theta}\,|f(\theta)|^2\,\mathrm{d}\theta\right\}=\int_{2\pi}\sin(\theta)|f(\theta+\theta_\circ)|^2\,\mathrm{d}\theta$$

vanishes, where $\mathrm{Im}\{\cdot\}$ denotes the imaginary part, and

$$\mathrm{Re}\left\{\mathrm{e}^{-j\theta_\circ}\int_{2\pi}\mathrm{e}^{j\theta}\,|f(\theta)|^2\,\mathrm{d}\theta\right\}=\int_{2\pi}\cos(\theta)|f(\theta+\theta_\circ)|^2\,\mathrm{d}\theta$$

is positive, where $\mathrm{Re}\{\cdot\}$ denotes the real part. Hence,

$$\theta_\circ=\arg\left\{\int_{2\pi}\mathrm{e}^{j\theta}\,|f(\theta)|^2\,\mathrm{d}\theta\right\}. \tag{13}$$

The possibility to find the optimum value of $\theta_\circ$ in such an easy way is definitely an advantage of the mapping $\xi(\theta)=2\sin(\theta/2)$ over the mapping $\xi(\theta)=2\tan(\theta/2)$. The value of $\theta_\circ$ in Eq. (13) turns out to correspond to the centroid proposed by Bandilla and Paul [2], and in fact it is easy to show that the width found through Eq. (4) by using this mapping also corresponds exactly to the one proposed by those authors, which can be interpreted in terms of the centroid of a ring [10,15].

3.3 Vertical projection

Our third mapping is

$$\xi(\theta)=\sin(\theta). \tag{14}$$

Like the previous mapping, this mapping has the property that the value $\theta_\circ$ can be found easily and in a way that is similar to the one described before. Using the same example of a rectangular function, $f(\theta)=\mathrm{rect}(\theta/2a\pi)$, we get

$$\sin(d_\theta)=\sqrt{[1-\sin(2a\pi)/(2a\pi)]/2}. \tag{15}$$

Again, for small a we have $d_\theta\simeq a\pi/\sqrt{3}$, while for $a\to 1$ we have $d_\theta\to\pi/4$. However, the same value $\pi/4$ also arises for $a=1/2$. This is caused, of course, by the fact that, unlike the other two mappings, the mapping $\xi(\theta)=\sin(\theta)$ is not monotonic in the interval $-\pi<\theta<\pi$, which makes it less attractive than the previous ones. Therefore, although not suitable for general functions, this mapping is appropriate if the function $f_\circ(\theta)$ is localized at $|\theta|\le\pi/2$.

3.4 Perimeter distance

The mapping

$$\xi(\theta)=\theta, \tag{16}$$

is a trivial unfolding of the circle onto a segment of the real axis. Of course, for the rectangular function, $f(\theta) = \mathrm{rect}(\theta/2a\pi)$, the width is the usual one: $d_\theta = a\pi/\sqrt{3}$. This mapping is included here because, to our knowledge, no uncertainty relation has yet been obtained involving only this width measure and the standard second–order moment in Fourier space.

3.5 Central projection

The fifth mapping considered here, given by

$$\xi(\theta) = \tan(\theta), \tag{17}$$

is perhaps the least convenient one. Like the vertical projection, the central projection is appropriate only for functions concentrated at $|\theta| \leq \pi/2$. We include it, however, because the results are trivially related to those for the stereographic projection. For example, the width of $f(\theta) = \mathrm{rect}(\theta/2a\pi)$ is given by

$$\tan(d_\theta) = \sqrt{\tan(a\pi)/(a\pi) - 1}. \tag{18}$$

Once more, for small a this reduces to $d_\theta \simeq a\pi/\sqrt{3}$. For $a \to 1/2$ on the other hand, we get $d_\theta \to \pi/2$.

4 Cauchy–Schwarz inequality

Before we start our treatment of periodic functions, we consider the Cauchy–Schwarz inequality that is used to derive the most elementary uncertainty relation for non–periodic functions,

$$\begin{aligned} \left|\int_{-\infty}^{+\infty} t f^*(t) f'(t)\,\mathrm{d}t\right|^2 &\leq \left(\int_{-\infty}^{+\infty} t^2|f(t)|^2\,\mathrm{d}t\right)\left(\int_{-\infty}^{+\infty} |f'(t)|^2\,\mathrm{d}t\right) \\ &= \left(\int_{-\infty}^{+\infty} t^2|f(t)|^2\,\mathrm{d}t\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2|F(\omega)|^2\,\mathrm{d}\omega\right), \end{aligned} \tag{19}$$

where $F(\omega)$ is the Fourier transform of $f(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t. \tag{20}$$

The important thing to realize is that the two operators “multiplication by t” and “differentiation to t” are related in the sense that “differentiation to t” of a function $f(t)$ corresponds to “multiplication by $j\omega$” of its Fourier transform

$F(\omega)$. A more general inequality would read

$$\left|\int_{-\infty}^{+\infty} \xi^*(t-t_\circ) f^*(t) \hat{f}(t)\,\mathrm{d}t\right|^2 \le \left(\int_{-\infty}^{+\infty} |\xi(t-t_\circ)|^2 |f(t)|^2\,\mathrm{d}t\right)\left(\int_{-\infty}^{+\infty} |\hat{f}(t)|^2\,\mathrm{d}t\right)$$
$$= \left(\int_{-\infty}^{+\infty} |\xi(t-t_\circ)|^2 |f(t)|^2\,\mathrm{d}t\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\eta(\omega-\omega_\circ)|^2 |F(\omega)|^2\,\mathrm{d}\omega\right), \quad (21)$$

where the function $\hat{f}(t)$ is defined via the Fourier transform domain by

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\eta(\omega-\omega_\circ) F(\omega)\,\mathrm{e}^{j\omega t}\,\mathrm{d}\omega. \quad (22)$$

Note in particular that we allow $\xi(t)$ and $\eta(t)$ to be complex.

Considering the left–hand side expression in Eq. (21), we might get the impression that the t domain and the ω domain are treated differently. This is not the case, however. If we define the function $\hat{F}(\omega)$ via the Fourier transform domain by

$$\hat{F}(\omega) = \int_{-\infty}^{+\infty} -j\xi(t-t_\circ) f(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t, \quad (23)$$

cf. Eq. (22), we can as well formulate this left–hand side expression in the ω domain, when we use Parseval's relation

$$\int_{-\infty}^{+\infty} \hat{f}(t)[\xi(t-t_\circ) f(t)]^*\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\eta(\omega-\omega_\circ) F(\omega)] \hat{F}^*(\omega)\,\mathrm{d}\omega, \quad (24)$$

and write the left–hand side of the Cauchy–Schwarz inequality (21) in the symmetric form

$$\left|\int_{-\infty}^{+\infty} \xi^*(t-t_\circ) f^*(t) \hat{f}(t)\,\mathrm{d}t\right| \left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} \eta^*(\omega-\omega_\circ) F^*(\omega) \hat{F}(\omega)\,\mathrm{d}\omega\right|. \quad (25)$$

The general form (21) of the Cauchy–Schwarz inequality reduces to the usual one (19) by choosing $t_\circ = \omega_\circ = 0$, $\xi(t) = t$ and hence $\hat{F}(\omega) = F'(\omega)$, and $\eta(\omega) = \omega$ and hence $\hat{f}(t) = f'(t)$.

We now use the general form of the Cauchy–Schwarz inequality, appropriately modified, to the case of periodic functions.

5 Uncertainty relations for periodic functions

We will now derive some uncertainty relations for periodic functions. To do so, we need a description of a periodic function $f(\theta)$ as a Fourier series with Fourier coefficients F_k:

$$f(\theta) = \sum_k F_k\,\mathrm{e}^{jk\theta} \qquad F_k = \frac{1}{2\pi}\int_{2\pi} f(\theta)\,\mathrm{e}^{-jk\theta}\,\mathrm{d}\theta. \quad (26)$$

The summation in the above expression is over all integers k and the integration is again over one arbitrary period 2π. In the context of this paper, Parseval's relation

$$\frac{1}{2\pi}\int_{2\pi} f(\theta)g^*(\theta)\,\mathrm{d}\theta = \sum_k F_k G_k^* \tag{27}$$

is relevant.

On the analogy of the Cauchy–Schwarz inequality (21) and with $\xi(\theta)$ a periodic mapping again, $\xi(\theta) = \xi(\theta + 2\pi)$, we get

$$\left|\frac{1}{2\pi}\int_{2\pi} \xi^*(\theta-\theta_\circ) f^*(\theta)\hat{f}(\theta)\,\mathrm{d}\theta\right|^2 \leq \left(\frac{1}{2\pi}\int_{2\pi} |\xi(\theta-\theta_\circ)|^2 |f(\theta)|^2\,\mathrm{d}\theta\right)\left(\sum_k |\eta(k-k_\circ)|^2 |F_k|^2\right). \tag{28}$$

The function $\hat{f}(\theta)$ and the sequence $\hat{F}_k$ are defined – via the Fourier transform domain – by

$$\hat{f}(\theta) = \sum_k j\eta(k-k_\circ)F_k\,\mathrm{e}^{jk\theta}, \tag{29}$$

$$\hat{F}_k = \frac{1}{2\pi}\int_{2\pi} -j\xi(\theta-\theta_\circ)f(\theta)\,\mathrm{e}^{-jk\theta}\,\mathrm{d}\theta, \tag{30}$$

cf. Eqs. (22) and (23), and from Parseval's relation, we have

$$\frac{1}{2\pi}\int_{2\pi} \hat{f}(\theta)[\xi(\theta-\theta_\circ)f(\theta)]^*\,\mathrm{d}\theta = \sum_k [\eta(k-k_\circ)F_k]\hat{F}_k^*, \tag{31}$$

cf. Eq. (24).

The mappings $\xi(\theta)$ and $\eta(k)$ and the values $\theta_\circ$ and $k_\circ$ have to be chosen appropriately in due time. Here we throughout choose $\eta(k) = k$, in which case $\hat{f}(\theta) = [f(\theta)\exp(-jk_\circ\theta)]'\exp(jk_\circ\theta)$. The mapping $\xi(\theta)$ may be chosen to be complex, in which case we will denote its absolute value by $\xi_\circ(\theta)$: $\xi_\circ(\theta) = |\xi(\theta)|$. For the expression that appears in the left–hand side of the inequality (28) we then have, using integration by parts,

$$\begin{aligned}&\int_{2\pi} \xi^*(\theta-\theta_\circ)f^*(\theta)\left[f(\theta)\,\mathrm{e}^{-jk_\circ\theta}\right]'\mathrm{e}^{jk_\circ\theta}\,\mathrm{d}\theta \\ &= -\int_{2\pi} \xi'^*(\theta-\theta_\circ)|f(\theta)|^2\,\mathrm{d}\theta - \int_{2\pi} \xi^*(\theta-\theta_\circ)\left\{f^*(\theta)\left[f(\theta)\,\mathrm{e}^{-jk_\circ\theta}\right]'\mathrm{e}^{jk_\circ\theta}\right\}^*\mathrm{d}\theta,\end{aligned} \tag{32}$$

and hence

$$\int_{2\pi} \xi^*(\theta-\theta_\circ)f^*(\theta)\left[f(\theta)\,\mathrm{e}^{-jk_\circ\theta}\right]'\mathrm{e}^{jk_\circ\theta}\,\mathrm{d}\theta = \tfrac{1}{2}A + \tfrac{1}{2}B, \tag{33}$$

with

$$A = -\int_{2\pi} \xi'^*(\theta-\theta_\circ)|f(\theta)|^2\,\mathrm{d}\theta,$$
$$B = -j\int_{2\pi} 2\xi^*(\theta-\theta_\circ)\,\mathrm{Im}\left\{f(\theta)\left[f^*(\theta)\,\mathrm{e}^{jk_\circ\theta}\right]'\mathrm{e}^{-jk_\circ\theta}\right\}\,\mathrm{d}\theta.$$

With the (positive) width parameter d_k defined as the normalized, (central) second–order moment of F_k, cf. Eq. (4),

$$\eta^2(d_k) = \frac{\sum_k \eta^2(k-k_\circ)|F_k|^2}{\sum_k |F_k|^2} = \frac{\sum_k (k-k_\circ)^2|F_k|^2}{\sum_k |F_k|^2} = d_k^2, \tag{34}$$

we then get the relationship

$$2\,\xi_\circ(d_\theta)\,d_k \geq \frac{|A+B|}{\int_{2\pi}|f(\theta)|^2\,\mathrm{d}\theta}. \tag{35}$$

Note that the width d_k takes its minimum value if $k_\circ$ is chosen as the normalized first–order moment of F_k, cf. Eq. (3).

If we would have chosen $\xi(\theta-\theta_\circ)$ instead of $\xi^*(\theta-\theta_\circ)$ in the above expressions, the left–hand side of the inequality (28) would have been equal to $\frac{1}{2}A^* - \frac{1}{2}B^*$, cf. Eq. (33), whereas the right–hand side would remain the same. We thus would get

$$2\,\xi_\circ(d_\theta)\,d_k \geq \frac{|A-B|}{\int_{2\pi}|f(\theta)|^2\,\mathrm{d}\theta}. \tag{36}$$

Since both Eq. (35) and Eq. (36) have to hold, we get

$$2\,\xi_\circ(d_\theta)\,d_k \geq \frac{\mathrm{Max}\{|A+B|,|A-B|\}}{\int_{2\pi}|f(\theta)|^2\,\mathrm{d}\theta} \geq \frac{|A|}{\int_{2\pi}|f(\theta)|^2\,\mathrm{d}\theta}. \tag{37}$$

Notice that the second inequality sign becomes an equality sign if $B=0$; this is the case, for instance, if $f(\theta)\exp(-jk_\circ\theta)$ has a constant phase.

Finally, we are led to the following uncertainty–type relationship:

$$2\,\xi_\circ(d_\theta)\,d_k \geq \frac{\left|\frac{1}{2\pi}\int_{2\pi}\xi'(\theta-\theta_\circ)|f(\theta)|^2 d\theta\right|}{\frac{1}{2\pi}\int_{2\pi}|f(\theta)|^2\,\mathrm{d}\theta} = \frac{\left|\frac{1}{2\pi}\int_{2\pi}\xi'(\theta)|f_\circ(\theta)|^2 d\theta\right|}{\frac{1}{2\pi}\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta}. \tag{38}$$

We remark that the derivative of $\xi(\theta)$ leads to Dirac pulses if $\xi(\theta)$ is not continuous. This is likely to happen for $\theta=\pi$, as we will see, in particular for the mappings $\xi_\circ(\theta)=\theta$ and $\xi_\circ(\theta)=2\sin(\theta/2)$. Fortunately, the possibility to have $\xi(\theta)$ complex by choosing a proper phase function will allow us to prevent the occurrence of a Dirac pulse at $\theta=\pi$.

For the mapping (7), $\xi(\theta) = 2\tan(\theta/2)$, the right–hand side of Eq. (38) takes the form

$$\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{1/\cos^2(\theta/2)\right\}|f_\circ(\theta)|^2\,\mathrm{d}\theta}{\frac{1}{2\pi}\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta} = 1 + \tan^2(d_\theta/2),$$

where, of course, the integrand in the numerator should behave properly in the neighborhood of π, i.e. $|f_\circ(\theta)|^2/\cos^2(\theta/2)$ should be integrable. The uncertainty–type relationship (38) can then be expressed as

$$4\,\tan(d_\theta/2)\,d_k \geq 1 + \tan^2(d_\theta/2), \tag{39}$$

or, after some straightforward calculations, as

$$2\,\sin(d_\theta)\,d_k \geq 1. \tag{40}$$

The latter relationship resembles the well–known uncertainty relation for non–periodic functions. This is not unexpected, since we have actually transformed the central period of the function $f_\circ(\theta) = f(\theta + \theta_\circ)$, $-\pi < \theta < \pi$, into a non–periodic function $g(\xi)$, say, that covers the entire ξ axis, $-\infty < \xi < \infty$.

Notice that there is an apparent problem with inequality (40): given that $\sin(d_\theta) \leq 1$, how can this relation be satisfied for $d_k < 1/2$? The answer is that, since we assumed that $f_\circ(\pi) = 0$, one cannot have complete localization in the Fourier transform domain, so for the functions for which inequality (40) is valid, $d_k \geq 1/2$.

It might be interesting to look for which functions the equality sign in the inequalities (39) and (40) would apply. From the inequality (28) we conclude that the equality sign applies if $\hat{f}_\circ(\theta) = f'_\circ(\theta) - jk_\circ f_\circ(\theta)$ is proportional to $\xi(\theta)f_\circ(\theta)$, or $d\ln f_\circ(\theta)/d\theta - jk_\circ$ is proportional to $\xi(\theta)$, and hence

$$f_\circ(\theta) = \exp\left[-\frac{2}{\sigma^2}\int_0^\theta \tan(\tau/2)\,\mathrm{d}\tau\right] = \left[\cos(\theta/2)\right]^{4/\sigma^2}, \tag{41}$$

where the proportionality constant was chosen as $-1/\sigma^2$. For convenience, we have omitted the modulating term that would arise in the case that $k_\circ \neq 0$. We remark that since $|f_\circ(\theta)|^2/\cos^2(\theta/2)$ has to remain integrable, we have the condition $0 < \sigma < 2\sqrt{2}$. These 'minimum uncertainty states' were also found in Ref. [7], where the same mapping was used. Note that for small σ we have $f_\circ(\theta) \simeq \exp(-\theta^2/2\sigma^2)$ which goes to zero very rapidly; in that case we have $d_\theta \simeq \sigma$ and $2d_\theta d_k \simeq 1$. In Fig. 2 we have depicted $f_\circ(\theta)$ for several values of σ and in Fig. 3 we have plotted d_θ as a function of σ.

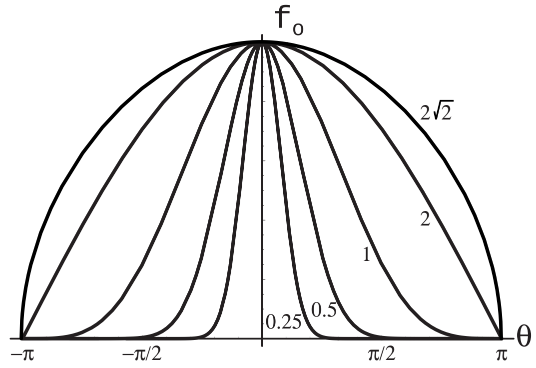

Fig. 2. Minimum uncertainty states for the width measure corresponding to the stereographic projection, for $\sigma = 0.25, 0.5, 1, 2, 2\sqrt{2}$.

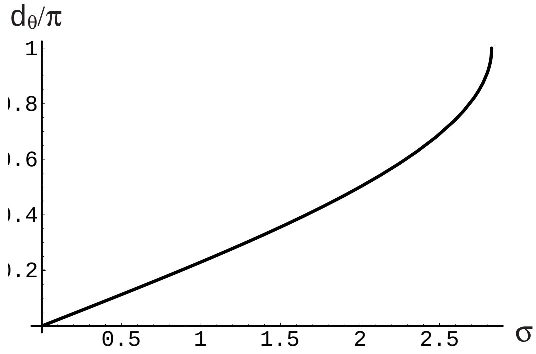

Fig. 3. Dependence of d_θ in σ for the minimum uncertainty states in Eq. (41).

For the *central* projection we find similar results. We only have to replace $2\tan(\theta/2)$ by $\tan\theta$, leading to the uncertainty–type relationship $\sin(2d_\theta)d_k \geq 1$, cf. Eq. (40), and to the minimum uncertainty state, cf. Eq. (41),

$$f_\circ(\theta) = \begin{cases} [\cos(\theta)]^{4/\sigma^2} & |\theta| \leq \pi/2 \\ 0 & \pi/2 < |\theta| \leq \pi, \end{cases} \tag{42}$$

with $0 < \sigma < 2\sqrt{2}$ again. These are the ground states of a harmonic oscillator analogue in a circular space as considered in Ref. [1]. Also, the Fourier coefficients found there for the function in Eq. (42) are consistent with those in Ref. [7] for the function in Eq. (41).

5.2 Cord and perimeter distances

The mapping (9), $\xi_\circ(\theta) = 2\sin(\theta/2)$, leads to similar results as the stereographic mapping, with the additional advantage of an easy way to determine the optimum value of $\theta_\circ$. We find the uncertainty–type relationship to be

$$4\,\sin(d_\theta/2)\,d_k \geq \frac{\left|\dfrac{1}{2\pi}\displaystyle\int_{-\pi^+}^{\pi^+}\{4\,\delta(\theta-\pi)-\cos(\theta/2)\}\,|f_\circ(\theta)|^2\,\mathrm{d}\theta\right|}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta}$$

$$= \frac{\dfrac{1}{2\pi}\left|4\,|f_\circ(\pi)|^2-\displaystyle\int_{-\pi}^{\pi}\cos(\theta/2)|f_\circ(\theta)|^2\,\mathrm{d}\theta\right|}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta}. \tag{43}$$

One way of avoiding the effect of the discontinuity of $\xi_\circ(\theta)$ at $\theta=\pi$ in Eq. (43) is to require that $f_\circ(\theta)$ is zero around that point, leading to

$$4\,\sin(d_\theta/2)\,d_k \geq \frac{\left|\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\cos(\theta/2)|f_\circ(\theta)|^2\,\mathrm{d}\theta\right|}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta}. \tag{44}$$

Although we cannot express the right–hand side of Eq. (44) in terms of the width d_θ, we can find a lower bound for it. Since $\cos(\theta/2)\geq\cos^2(\theta/2)=1-\sin^2(\theta/2)$ on the interval $-\pi<\theta<\pi$, we have

$$4\,\sin(d_\theta/2)\,d_k \geq 1-\sin^2(d_\theta/2)$$

or

$$4\,\frac{\sin(d_\theta/2)}{1-\sin^2(d_\theta/2)}\,d_k \geq 1. \tag{45}$$

We are also tempted to say that the equality sign in Eq. (44) is reached for

$$f_\circ(\theta)=\exp\left[-\frac{2}{\sigma^2}\int_0^\theta \sin(\tau/2)\,\mathrm{d}\tau\right]=\exp\left[-4\{1-\cos(\theta/2)\}/\sigma^2\right], \tag{46}$$

but this is not the case, since this function $f_\circ(\theta)$ is not zero around $\theta=\pi$.

It is desirable to find an inequality that involves no knowledge of the function $f_\circ(\theta)$ other than d_θ and d_k. [Remember that the inequality (43) involves $f_\circ(\pi)$.] This can be done by removing the discontinuity of $\xi_\circ(\theta)$ at $\theta=\pi$ by adding a phase factor $\exp(j\theta/2)$,

$$\xi(\theta)=\xi_\circ(\theta)\,\mathrm{e}^{j\theta/2}=2\,\sin(\theta/2)\,\mathrm{e}^{j\theta/2}=\sin\theta+j(1-\cos\theta), \tag{47}$$

and hence $\xi'(\theta)=\cos\theta+j\sin\theta$. Proceeding as before, and noting that $\theta_\circ$ has been chosen such that

$$\int_{2\pi}\sin(\theta)|f_\circ(\theta)|^2\,\mathrm{d}\theta=0,$$

$$\int_{2\pi}\cos(\theta)|f_\circ(\theta)|^2\,\mathrm{d}\theta>0,$$

see Eqs. (11) and (12), we get the uncertainty–type relationship

$$4\,\sin(d_\theta/2)\,d_k \geq \frac{\dfrac{1}{2\pi}\displaystyle\int_{2\pi} \cos(\theta)|f_\circ(\theta)|^2\,\mathrm{d}\theta}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi} |f_\circ(\theta)|^2\,\mathrm{d}\theta} = |1-2\sin^2(d_\theta/2)|, \qquad (48)$$

cf. Eq. (44), or

$$4\,\frac{\sin(d_\theta/2)}{|1-2\sin^2(d_\theta/2)|}\,d_k = 4\,\frac{\sin(d_\theta/2)}{\cos(d_\theta)}\,d_k \geq 1, \qquad (49)$$

cf. Eq. (45). The absolute value bars in (49) can be removed because, as it is easy to show, $0 \leq d_\theta \leq \pi/2$.

The inequality in Eq. (49) turns out to be a bit weaker than the one found, for this same width measure, in Refs. [10,14,15]:

$$2\,\tan(d_\theta)\,d_k \geq 1. \qquad (50)$$

As it turns out, even this stronger relation is not as strong as possible (i.e. the equality sign cannot be reached except for $d_\theta = 0$). In order to find the true bounds for d_θ and d_k, one must follow an alternative route, based on the eigenfunctions of an appropriate operator [12,16]. It is shown in these references that, while the bounds are not expressible in simple algebraic terms, the minimum uncertainty states turn out to be Mathieu functions.

For the case of the perimeter distance mapping, one might expect to obtain the usual uncertainty relation $2d_\theta d_k \geq 1$. However, because of the discontinuity of the mapping, one obtains an inequality like (43):

$$2\,d_\theta\,d_k \geq \left|\frac{|f_\circ(\pi)|^2}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi} |f_\circ(\theta)|^2\,\mathrm{d}\theta} - 1\right|. \qquad (51)$$

Therefore, we only recover the usual uncertainty relation for functions with $f_\circ(\pi) = 0$. The inequality (51) was found independently by several groups [5,13,17].

As with $2\sin(\theta/2)$, we can remove the discontinuity that arises for the mapping $\xi_\circ(\theta) = \theta$ by adding a phase factor. An appropriate factor for this case is $\exp[j\arcsin\{(\theta/\pi)[2-(\theta/\pi)^2]^{1/2}\}] = 1-(\theta/\pi)^2 + j(\theta/\pi)[2-(\theta/\pi)^2]^{\frac{1}{2}}$, so $\mathrm{Re}\,\{\xi'(\theta)\} = 1 - 3\,\xi_\circ^2(\theta)/\pi^2$ where $\mathrm{Re}\{\cdot\}$ denotes the real part, leading to the relation

$$2\,\frac{d_\theta}{1-3(d_\theta/\pi)^2}\,d_k \geq 1. \qquad (52)$$

Since we ignored the imaginary part of ξ', we expect that inequality (52) cannot be satisfied as an equality except in trivial limits. To our knowledge, however, this is the strictest inequality that has been found for the case when the function's width is measured as a minimized standard deviation over a one–period window and the conjugate width is given by the standard deviation of the Fourier coefficients.

5.3 Vertical projection

For the mapping (14), $\xi(\theta) = \sin(\theta)$, we find similar results. Since $\xi(\theta)$ does not have discontinuities, there is no need for an additional phase factor. The uncertainty relation now reads

$$2\,\sin(d_\theta)\,d_k \ge \frac{\left|\dfrac{1}{2\pi}\displaystyle\int_{2\pi}\cos(\theta)|f_\circ(\theta)|^2\,\mathrm{d}\theta\right|}{\dfrac{1}{2\pi}\displaystyle\int_{2\pi}|f_\circ(\theta)|^2\,\mathrm{d}\theta}, \tag{53}$$

where the lower bound is the same as that in Eq. (48), and the equality sign is reached for

$$f_\circ(\theta) = \exp\left[-\frac{1}{\sigma^2}\int_0^\theta \sin(\tau)\,\mathrm{d}\tau\right] = \exp\left[-\{1-\cos(\theta)\}/\sigma^2\right]. \tag{54}$$

Note again that for small θ we have $f_\circ(\theta) \simeq \exp(-\theta^2/2\sigma^2)$ and that for small σ the function $f_\circ(\theta)$ will go to zero very rapidly; in that case we have $d_\theta \simeq \sigma$ and $2d_\theta d_k \simeq 1$. The restriction to small σ has the advantage that the function $f_\circ(\theta)$ is roughly restricted to the domain $|\theta| \le \pi/2$, in which case d_θ is indeed a proper measure of the width. In Fig. 4 we have depicted $f_\circ(\theta)$ for several values of σ and in Fig. 5 we have plotted d_θ as a function of σ.

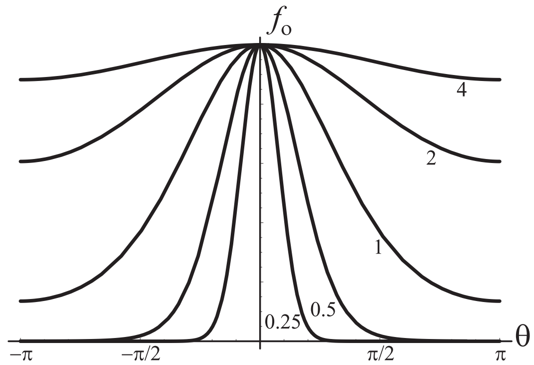

Fig. 4. Minimum uncertainty states for the width measure corresponding to the vertical projection, for $\sigma = 0.25, 0.5, 1, 2, 4$.

If the domain of $f_\circ(\theta)$ is restricted to $|\theta| < \pi/2$, we can write again $\cos(\theta) \ge \cos^2(\theta) = 1 - \sin^2(\theta)$, and the lower bound can be approximated as we did in

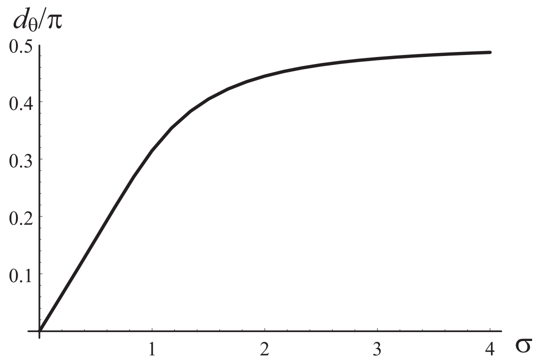

Fig. 5. Dependence of d_θ in σ for the minimum uncertainty states in Eq. (54).

the previous section, leading to the relation

$$2\,\frac{\sin(d_\theta)}{1-\sin^2(d_\theta)}\,d_k \geq 1, \tag{55}$$

cf. Eq. (49).

We finally remark that for small values of d_θ, all uncertainty relations that have been presented in Section 5 reduce to $2d_\theta d_k \geq 1$, and become similar to the well–known uncertainty relation for non–periodic functions $2d_t d_\omega \geq 1$.

6 Conclusion

We have presented a generalized framework for obtaining uncertainty relations between a periodic function and its Fourier series, corresponding to several measures for the width of the function and to the standard second–order moment of the Fourier coefficients. In particular, we investigated five different width measures of a periodic function, associated with five simple mappings between a circle and a straight line or line segment. Several of the resulting measures have been used previously, although the corresponding uncertainty relations were not known. In particular, the minimized standard deviation over a period (corresponding to the perimeter mapping) is perhaps the most straightforward width definition for a periodic function. Yet, no uncertainty relation involving only the widths considered here had been found before, to our knowledge. All previously found relations involved either extra information about the function [5,13,17], or used a non-standard width measure in Fourier space [8]. The approach taken here, which involved including a suitable phase factor, yielded a simple, intuitive relation. This relation is, nevertheless, unachievable as an equality. The mapping approach might also be applicable to the case of periodic, discrete sequences and their discrete Fourier transforms. This, however, falls beyond the scope of this paper.

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References

[1] M. A. Alonso, G. S. Pogosyan, and K. B. Wolf, "Wigner functions for curved spaces. II On spheres," *J. Math. Phys.*, Vol. **44**, 2003, pp. 1472–1489

[2] A. Bandilla and H. Paul, "Laser amplifier and phase uncertainty," *Ann. Phys (Leipzig)*, Vol. **23**, 1969, pp. 323–333

[3] I. Bialynicki–Birula, M. Freyberger, and W. Schleich, "Various measures of quantum phase uncertainty: a comparative study," *Physica Scripta*, Vol. **T48**, 1993, pp. 113–118

[4] N. G. de Bruijn, "Uncertainty principles in Fourier analysis," in: O. Shisha, ed., *Inequalities*, Academic, New York, 1967, pp. 57–71

[5] L. Calvez and P. Vilbé, "On the uncertainty principle in discrete signals," *IEEE Trans. Circuits Syst.*, Vol. **39**, 1992, pp. 394–395

[6] L. Cohen, *Time–Frequency Analysis*, Prentice–Hall, Englewood Cliffs, NJ, 1995, Chapter 3

[7] M. I. Doroslovački, "Product of second moments in time and frequency for discrete–time signals and the uncertainty limit," *Signal Process.*, Vol. **67**, 1998, pp. 59–76

[8] M. I. Doroslovački, "Uncertainty relations involving ordinary second moments in time and frequency for discrete–time signals," *Signal Process.*, Vol. **81**, 2001, pp. 409-418

[9] G. Folland and A. Sitaram, "The uncertainty principle: a mathematical survey," *J. Fourier Anal. Appl.*, Vol. **3**, 1997, pp. 207–238

[10] G. W. Forbes and M. A. Alonso, "Measures and spread for periodic distributions and the associated uncertainty relations," *Am. J. Phys.*, Vol. **69**, 2001, pp. 340–347

[11] G. W. Forbes and M. A. Alonso, "Consistent analogs of the Fourier uncertainty relation," *Am. J. Phys.*, Vol. **69**, 2001, pp. 1091–1095

[12] G. W. Forbes, M. A. Alonso, and A. E. Siegman, “Uncertainty relations and minimum–uncertainty states for the discrete Fourier transform and the Fourier Series,” *J. Phys. A*, Vol. **36**, 2003, pp. 7027–7047.

[13] R. Ishii and K. Furukawa, “The uncertainty principle in discrete signals,” *IEEE Trans. Circuits Syst.*, Vol. **33**, 1986, pp. 1032–1034

[14] A. Lukš and V. Peřinová, “Extended number state basis and number–phase intelligent states of light fields. I. Mapping and operator ordering approach to quantum phase probelms,” *Czech. J. Phys.*, Vol. **41**, 1991, pp. 1205–1230

[15] T. Opatrný, “Mean value and uncertainty in optical phase – a simple mechanical analogy,” *J. Phys. A*, Vol. **27**, 1994, pp. 7201–7208

[16] T. Opatrný, “Number–phase uncertainty relations,” *J. Phys. A*, Vol. **28**, 1995, pp. 6961–6975

[17] D. T. Pegg and S. M. Barnett, “Phase properties of the quantized single–mode electromagnetic field,” *Phys. Rev. A*, Vol. **39**, 1989, pp. 1665–1675

[18] D. T. Pegg and S. M. Barnett, “Tutorial review: Quantum optical phase,” *J. Mod. Optics*, Vol. **44**, 1997, pp. 225–264