Wigner distribution moments in fractional Fourier transform systems

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It is shown how all global Wigner distribution moments of arbitrary order in the output plane of a (generally anamorphic) two-dimensional fractional Fourier transform system can be expressed in terms of the moments in the input plane. Since Wigner distribution moments are identical to derivatives of the ambiguity function at the origin, a similar relation holds for these derivatives. The general input-output relationship is then broken down into a number of rotation-type input-output relationships between certain combinations of moments. It is shown how the Wigner distribution moments (or ambiguity function derivatives) can be measured as intensity moments in the output planes of a set of appropriate fractional Fourier transform systems and thus be derived from the corresponding fractional power spectra. The minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments is derived. As an important by-product we get a number of moment combinations that are invariant under (anamorphic) fractional Fourier transformation.

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1. Introduction

After the introduction of the Wigner distribution[1] (WD) for the description of coherent and partially coherent optical fields,[2] it became an important tool for optical signal/image analysis and beam characterization.[3–6] The WD completely describes the complex amplitude of a coherent optical field (up to a constant phase factor) or the two-point correlation function of a partially coherent field. Since the WD of a two-dimensional optical field is a function of four variables, it is difficult to analyze. Therefore, the optical field is often represented not by the WD itself, but by its global moments. Beam characterization based on the second-order moments of the WD thus became the basis of an International Organization for Standardization standard.[7]

Some of the WD moments can directly be determined from measurements of the intensity distributions in the image plane or the Fourier plane, but most of the moments cannot be determined in such an easy way. In order to calculate such moments, additional information is required. Since first-order optical systems[8] – also called $ABCD$ systems – produce affine transformations of the WD in phase space, the intensity distributions measured at the output of such systems can provide such additional information. It was shown[9,10] that for these purposes, $ABCD$ systems using cylindrical lenses (anamorphic $ABCD$

systems) have to be applied. The application of $ABCD$ systems for the measurements of the second-order WD moments have been reported in several publications.[9–14]

It is the aim of this paper to show how all WD moments can be measured as intensity moments only; and since WD moments are identical to derivatives of the ambiguity function[15–17] at the origin, the same holds for these derivatives. We therefore consider a particular case of the $ABCD$ system: the anamorphic fractional Fourier transform (FT) system. We show that not only the second-order moments, but all moments of the four-dimensional WD can be obtained from measurements of only intensity distributions in an appropriate number of (anamorphic) fractional FT systems. The minimum number of (anamorphic) fractional power spectra needed for the calculation of all WD moments of an arbitrary order is determined. As a by-product, moment combinations are found that are invariant under anamorphic fractional Fourier transformation.

2. Wigner distribution and ambiguity function

The Wigner distribution (WD) of a twodimensional function $f(x, y)$ is defined by

$$W_f(x,u;y,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tfrac{1}{2}x', y+\tfrac{1}{2}y')\, f^*(x-\tfrac{1}{2}x', y-\tfrac{1}{2}y')\exp[-j2\pi(ux'+vy')]dx'dy'. \quad (1)$$

The WD $W_f(x,u;y,v)$ represents a space function $f(x, y)$ in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x, and v the spatial-frequency variable associated to the space variable y. We remark that the definition of the WD – and all the results of this paper – need not be restricted to coherent light, in which case $f(x, y)$ would represent the complex field amplitude of the light, but can be extended to partially coherent light, in which case the two-point correlation function of the light can be identified with $< f(x+\frac{1}{2}x', y+\frac{1}{2}y')$ $f^*(x-\frac{1}{2}x', y-\frac{1}{2}y') >$.

The WD is related to the ambiguity function (AF) defined by

$$A_f(x',u';y',v') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tfrac{1}{2}x', y+\tfrac{1}{2}y')\, f^*(x-\tfrac{1}{2}x', y-\tfrac{1}{2}y')\exp[-j2\pi(u'x+v'y)]dxdy. \quad (2)$$

The relationship between the WD and the AF takes the form of a combined forward/inverse Fourier transformation:

$$A_f(x',u';y',v') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)\exp[-j2\pi(u'x-ux'+v'y-vy')]dxdudydv. \quad (3)$$

In this paper we consider the normalized moments of the WD (or the derivatives of the AF at the origin), where the normalization is with respect to the total energy E of the signal:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)dxdudydv = A_f(0,0;0,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dxdy. \quad (4)$$

These normalized moments μ_{pqrs} of the WD (or derivatives of the AF) are thus defined by

$$\begin{aligned}
\mu_{pqrs}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)x^p u^q y^r v^s dxdudydv && (5)\\
&= \frac{(-1)^{p+r}}{(j2\pi)^{p+q+r+s}}\,\frac{\partial^{p+q+r+s}A_f(x,u;y,v)}{\partial x^q\partial u^p\partial y^s\partial v^r}\bigg|_{x=u=y=v=0} && (6)\\
&= \frac{1}{(j2\pi)^{q+s}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r \\
&\quad\times \frac{\partial^{q+s}}{\partial\xi^q\partial\eta^s} f(x+\tfrac{1}{2}\xi, y+\tfrac{1}{2}\eta)\, f^*(x-\tfrac{1}{2}\xi, y-\tfrac{1}{2}\eta)\bigg|_{\xi=\eta=0} dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(j4\pi)^{q+s}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r \\
&\quad \times \left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^q \left(\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}\right)^s f(x_1,y_1)\, f^*(x_2,y_2)\Big|_{x_1=x_2=x,\, y_1=y_2=y} dx\,dy,
\end{aligned}
$$

with $p, q, r, s \geq 0$. Note that for $q = s = 0$ we have intensity moments, which can easily be measured:

$$
\begin{aligned}
\mu_{p0r0}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)\, x^p y^r\, dx\,du\,dy\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r |f(x,y)|^2 dx\,dy, \qquad (7)
\end{aligned}
$$

with $p, r \geq 0$.

We remark that the moments of other members $C_f(x,u;y,v)$ of the so-called Cohen class of phase-space representations,[18] which are defined by

$$
C_f(x,u;y,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x_o,u_o;y_o,v_o)\, W_f(x-x_o,u-u_o;y-y_o,v-v_o)\, dx_o\,du_o\,dy_o\,dv_o, \quad (8)
$$

are directly related to the WD moments μ_{pqrs}. To show this, it is advantageous to go from the Wigner-domain representation [see Eq. (8)] to the ambiguity-domain representation

$$
\bar{C}_f(x',u';y',v') = \bar{\Phi}(x',u';y',v')\, A_f(x',u';y',v'), \qquad (9)
$$

where $\bar{C}_f(x',u';y',v')$ and $\bar{\Phi}(x',u';y',v')$ represent the combined forward/inverse Fourier transforms of $C_f(x,u;y,v)$ and $\Phi(x,u;y,v)$, respectively, defined in a way similar to Eq. (3), and where the convolution (8) has been replaced by a simple product of $\bar{\Phi}(x',u';y',v')$ and the ambiguity function $A_f(x',u';y',v')$. Note that there is a wealth of phase-space representations[19] that belong to the Cohen class, each characterized by its own kernel $\Phi(x,u;y,v)$ or $\bar{\Phi}(x',u';y',v')$; the WD itself arises for $\bar{\Phi}(x',u';y',v') = 1$, of course, while the kernels $\bar{\Phi}(x',u';y',v')$ of the Rihaczek distribution, the pseudo WD [with window function $\eta(\frac{1}{2}x, \frac{1}{2}y)$], and the spectrogram [with window function $\gamma(x,y)$] – to mention only a few well-known phase-space representations – read $\exp[j\pi(x'u'+y'v')]$, $\eta(\frac{1}{2}x', \frac{1}{2}y')\eta^*(-\frac{1}{2}x', -\frac{1}{2}y')$, and $A_\gamma(-x',-u';-y',-v')$, respectively.[19]

The moments γ_{pqrs} of an arbitrary Cohen-class phase-space representation $C_f(x,u;y,v)$ now take the form [cf. Eqs. (5) and (6)]

$$
\begin{aligned}
\gamma_{pqrs} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_f(x,u;y,v)\, x^p u^q y^r v^s\, dx\,du\,dy\,dv \\
&= \frac{(-1)^{p+r}}{(j2\pi)^{p+q+r+s}} \frac{\partial^{p+q+r+s}\bar{C}_f(x,u;y,v)}{\partial x^q \partial u^p \partial y^s \partial v^r}\Big|_{x=u=y=v=0} \\
&= \frac{(-1)^{p+r}}{(j2\pi)^{p+q+r+s}} \frac{\partial^{p+q+r+s}\bar{\Phi}(x,u;y,v) A_f(x,u;y,v)}{\partial x^q \partial u^p \partial y^s \partial v^r}\Big|_{x=u=y=v=0} \\
&= \frac{(-1)^{p+r}}{(j2\pi)^{p+q+r+s}} \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s} \binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} \\
&\quad \times \frac{\partial^{k+l+m+n}\bar{\Phi}(x,u;y,v)}{\partial x^l \partial u^k \partial y^n \partial v^m} \frac{\partial^{p+q+r+s-k-l-m-n} A_f(x,u;y,v)}{\partial x^{q-l} \partial u^{p-k} \partial y^{s-n} \partial v^{r-m}}\Big|_{x=u=y=v=0} \\
&= E \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s} \binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} \\
&\quad \times \frac{(-1)^{k+m}}{(j2\pi)^{k+l+m+n}} \frac{\partial^{k+l+m+n}\bar{\Phi}(x,u;y,v)}{\partial x^l \partial u^k \partial y^n \partial v^m}\Big|_{x=u=y=v=0} \mu_{p-k,q-l,r-m,s-n} \qquad (10)
\end{aligned}
$$

Note that if $\partial^{k+l+m+n}\bar{\Phi}/\partial x^l \partial u^k \partial y^n \partial v^m|_{x=u=y=v=0}$ vanishes for $k = 1, \ldots, p$, $l = 1, \ldots, q$, $m = 1, \ldots, r$, and $n = 1, \ldots, s$, then the moments of the general phase-space representation $C_f(x, u; y, v)$ up to the order $(p + q + r + s)$, are identical to those of the WD. Furthermore, for the important special class of product kernels $\bar{\Phi}(x', u'; y', v') = S_x(x'u')S_y(y'v')$ [with examples the (real-valued) generalized Wigner, the Rihaczek, the Born-Jordan, the Choi-Williams, and the reduced interference kernels[19]], we get

$$\left.\frac{\partial^{2k+2m} S_x(xu)S_y(yv)}{\partial x^k \partial u^k \partial y^m \partial v^m}\right|_{x=u=y=v=0} = \left.\frac{\partial^{2k} S_x(xu)}{\partial x^k \partial u^k}\right|_{x=u=0} \left.\frac{\partial^{2m} S_y(yv)}{\partial y^m \partial v^m}\right|_{y=v=0} = k!m!\; S_x^{(k)}(0) S_y^{(m)}(0) \tag{11}$$

while all other derivatives vanish; in that case Eq. (10) reduces to the simpler form

$$\gamma_{pqrs} = E \sum_{k=0}^{\min(p,q)} \sum_{m=0}^{\min(r,s)} \binom{p}{k}\binom{q}{k}\binom{r}{m}\binom{s}{m} (2\pi)^{-(2k+2m)} k!m!\; S_x^{(k)}(0) S_y^{(m)}(0) \mu_{p-k,q-k,r-m,s-m}. \tag{12}$$

We conclude that knowledge of the WD moments (or the AF derivatives at the origin) is of vital importance for all kinds of phase-space signal representations. In this paper we will derive how these moments (and derivatives) are related to the moments of the fractional FT, and how they can be measured as intensity moments in the fractional FT domain.

3. Fractional Fourier transform

The (anamorphic) fractional Fourier transform (FT) of a function $f(x, y)$ is defined by[20–24]

$$F_{\alpha\beta}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_\alpha(x, u) K_\beta(y, v) f(x, y) dx dy, \tag{13}$$

where the kernel $K_\alpha(x, u)$ is given by

$$K_\alpha(x, u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j \sin\alpha}} \exp\left[j\pi \frac{(x^2 + u^2)\cos\alpha - 2ux}{\sin\alpha} \right]. \tag{14}$$

We remark that $F_{0,0}(u, v) = f(u, v)$ represents the function itself, while $F_{\pi/2,\pi/2}(u, v)$ corresponds to the normal two-dimensional FT of the function $f(x, y)$. Note, moreover, that $K_{\alpha+\pi}(x, u) = K_\alpha(x, -u)$ and hence $F_{\alpha+\pi,\beta+\pi}(u, v) = F_{\alpha\beta}(-u, -v)$, that $K_\alpha(x, u) = K_\alpha(u, x)$, and that $K_\alpha^*(x, u) = K_{-\alpha}(x, u)$.

The fractional FT can be generated optically by very simple, anamorphic, coherent-optical set-ups,[20] see Fig. 1, consisting only of cylindrical lenses whose focal lengths – in combination with some appropriate sections of free space – are related to the angles α and β. If we would choose the configuration in Fig. 1(a), for instance, we would have a distance $2d$ between the input and the output plane, and a middle plane in which two cylindrical lenses are located, with their axes oriented perpendicular to each other: one lens acting for the x coordinate (with its focal length f_x determined by α through $d = 2 f_x \sin^2 \frac{1}{2}\alpha$) and the other one (with its focal length f_y determined by β through $d = 2 f_y \sin^2 \frac{1}{2}\beta$) acting for the y coordinate. We could also choose different configurations for the x and y directions, of course. The input-output relationship of such anamorphic fractional FT systems in terms of the so-called ray transformation matrix (or $ABCD$

Figure 1 somewhere here

Fig. 1.

matrix),[8] where the positions x_i, y_i and directions u_i, v_i of an optical ray in the input plane are related to the positions x_o, y_o and directions u_o, v_o of the ray in the output plane, takes the form

$$\begin{bmatrix} x_o \\ y_o \\ u_o \\ v_o \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & B_{xu} & B_{xv} \\ A_{yx} & A_{yy} & B_{yu} & B_{yv} \\ C_{ux} & C_{uy} & D_{uu} & D_{uv} \\ C_{vx} & C_{vy} & D_{vu} & D_{vv} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\beta & 0 & \sin\beta \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ u_i \\ v_i \end{bmatrix} \tag{15}$$

or, with a different ordering of the variables,

$$\begin{bmatrix} x_o \\ u_o \\ y_o \\ v_o \end{bmatrix} = \begin{bmatrix} A_{xx} & B_{xu} & A_{xy} & B_{xv} \\ C_{ux} & D_{uu} & C_{uy} & D_{uv} \\ A_{yx} & B_{yu} & A_{yy} & B_{yv} \\ C_{vx} & D_{vu} & C_{vy} & D_{vv} \end{bmatrix} \begin{bmatrix} x_i \\ u_i \\ y_i \\ v_i \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x_i \\ u_i \\ y_i \\ v_i \end{bmatrix}. \tag{16}$$

One of the most important properties of the fractional FT is that it corresponds to a rotation of the WD and the AF in the phase space:

$$\begin{aligned} W_{F_{\alpha\beta}}(x,u;y,v) &= W_f(x\cos\alpha - u\sin\alpha, x\sin\alpha + u\cos\alpha; \\ &\qquad y\cos\beta - v\sin\beta, y\sin\beta + v\cos\beta), \qquad (17) \\ A_{F_{\alpha\beta}}(x',u';y',v') &= A_f(x'\cos\alpha - u'\sin\alpha, x'\sin\alpha + u'\cos\alpha; \\ &\qquad y'\cos\beta - v'\sin\beta, y'\sin\beta + v'\cos\beta). \qquad (18) \end{aligned}$$

Moreover, the fractional power spectra, i.e., the squared modulus $|F_{\alpha\beta}(x,y)|^2$ of the fractional Fourier transform, is directly related to the WD through a projection operation, and to the AF through an inverse Fourier transformation:

$$\begin{aligned} \left|F_{\alpha\beta}(x,y)\right|^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v)\,du\,dv, \qquad (19) \\ \left|F_{\alpha\beta}(x,y)\right|^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{F_{\alpha\beta}}(0,u';0,v')\exp[j2\pi(u'x+v'y)]\,du'\,dv'. \qquad (20) \end{aligned}$$

Note that the fractional power spectrum is related to the intensity distribution in the output plane of an anamorphic fractional FT system [see Eqs. (15) and (16)] and therefore can easily be measured in experiments.

In the remainder of the paper we will concentrate on the WD and its moments; the results for the AF and its derivatives at the origin are identical, see Eqs. (5) and (6).

4. Moments in the fractional domain

On the analogy of Eq. (5), we now define normalized moments $\mu_{pqrs}(\alpha,\beta)$ in the fractional domain and relate these to the original moments $\mu_{pqrs} = \mu_{pqrs}(0,0)$:

$$\begin{aligned} \mu_{pqrs}(\alpha,\beta)E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v)x^p u^q y^r v^s\,dx\,du\,dy\,dv \qquad (21) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x\cos\alpha - u\sin\alpha, x\sin\alpha + u\cos\alpha; \\ &\qquad y\cos\beta - v\sin\beta, y\sin\beta + v\cos\beta)x^p u^q y^r v^s\,dx\,du\,dy\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)(x\cos\alpha + u\sin\alpha)^p(-x\sin\alpha + u\cos\alpha)^q \\ &\qquad \times\ (y\cos\beta + v\sin\beta)^r(-y\sin\beta + v\cos\beta)^s\,dx\,du\,dy\,dv, \end{aligned}$$

with $p, q, r, s \geq 0$. The general relationship thus takes the form

$$\begin{aligned}\mu_{pqrs}(\alpha, \beta) &= \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s}\binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n}(-1)^{l+n}\mu_{p-k+l,q-l+k,r-m+n,s-n+m} \\ &\quad \times (\cos\alpha)^{p-k+q-l}(\sin\alpha)^{k+l}(\cos\beta)^{r-m+s-n}(\sin\beta)^{m+n}, \qquad (22)\end{aligned}$$

and for the intensity moments in particular we have

$$\mu_{p0r0}(\alpha, \beta) = \sum_{k=0}^{p}\sum_{m=0}^{r}\binom{p}{k}\binom{r}{m}\mu_{p-k,k,r-m,m}\cos^{p-k}\alpha\sin^{k}\alpha\cos^{r-m}\beta\sin^{m}\beta. \qquad (23)$$

Note that the total energy E, see Eq. (4), is invariant under fractional Fourier transformation:

$$\begin{aligned}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x, u; y, v)dxdudydv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f}(x, u; y, v)dxdudydv. \qquad (24)\end{aligned}$$

From the definitions of the normalized moments in the fractional domain, it follows directly that the $x^p u^q$ moments (with $r = s = 0$) are not affected by the fractional Fourier transformation in the y-direction (with angle β), while the $y^r v^s$ moments (with $p = q = 0$) are not affected by the one in the x-direction (with angle α):

$$\begin{aligned}\mu_{pq00}(\alpha, \beta) &= \mu_{pq00}(\alpha, 0), \\ \mu_{00rs}(\alpha, \beta) &= \mu_{00rs}(0, \beta).\end{aligned} \qquad (25)$$

Moreover, the $x^p y^r$ moments (with $q = s = 0$) can easily be measured as intensity moments again,

$$\mu_{p0r0}(\alpha, \beta)E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r |F_{\alpha\beta}(x, y)|^2 dxdy, \qquad (26)$$

and from the relationship

$$\mu_{pqrs}(\alpha, \beta) = (-1)^{p+r}\mu_{qpsr}(\alpha + \tfrac{1}{2}\pi, \beta + \tfrac{1}{2}\pi), \qquad (27)$$

we conclude that the $u^q v^s$ moments (with $p = r = 0$) can be measured as intensity moments, as well.

5. Relations between moments in the fractional domain

To find simple relationships between the moments of $W_f(x, u; y, v)$ on the one hand and those of $W_{F_{\alpha,\beta}}(x, u; y, v)$ on the other, cf. Eq. (21), we define[25]

$$\begin{aligned}\xi &= x + ju, \\ \eta &= y + jv,\end{aligned} \qquad (28)$$

$$\begin{aligned}\xi(\alpha) &= \exp(-j\alpha)\xi, \\ \eta(\beta) &= \exp(-j\beta)\eta,\end{aligned} \qquad (29)$$

and

$$\begin{aligned}\xi_{2k,l}(\alpha) &= [\xi(\alpha)]^{k+l}[\xi^*(\alpha)]^k &&= |\xi(\alpha)|^{2k}[\xi(\alpha)]^l, \\ \eta_{2m,n}(\beta) &= [\eta(\beta)]^{m+n}[\eta^*(\beta)]^n &&= |\eta(\beta)|^{2m}[\eta(\alpha)]^n,\end{aligned} \qquad (30)$$

with $k, l \geq 0$ and $m, n \geq 0$. With $\xi_{2k,l}(0) = \xi_{2k,l}$ and $\eta_{2m,n}(0) = \eta_{2m,n}$, we then have

$$\begin{aligned} \xi_{2k,l}(\alpha)\eta_{2m,n}(\beta) &= \exp[-j(l\alpha + n\beta)]\xi_{2k,l}\eta_{2m,n}, \\ \xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta) &= \exp[-j(l\alpha - n\beta)]\xi_{2k,l}\eta^*_{2m,n}, \end{aligned} \tag{31}$$

which equations are equivalent to the rotation operators

$$\begin{aligned} \begin{bmatrix} \Re\{\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)\} \\ \Im\{\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)\} \end{bmatrix} &= \mathbf{R}(l\alpha + n\beta) \begin{bmatrix} \Re\{\xi_{2k,l}\eta_{2m,n}\} \\ \Im\{\xi_{2k,l}\eta_{2m,n}\} \end{bmatrix}, \\ \begin{bmatrix} \Re\{\xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta)\} \\ \Im\{\xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta)\} \end{bmatrix} &= \mathbf{R}(l\alpha - n\beta) \begin{bmatrix} \Re\{\xi_{2k,l}\eta^*_{2m,n}\} \\ \Im\{\xi_{2k,l}\eta^*_{2m,n}\} \end{bmatrix}, \end{aligned} \tag{32}$$

respectively, where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and the imaginary part, respectively, and where $\mathbf{R}(\alpha)$ represents the rotation matrix

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \tag{33}$$

corresponding to a clockwise rotation through an angle α. The real and imaginary parts of $\xi_{2k,l}\eta_{2m,n}$ (and of $\xi_{2k,l}\eta^*_{2m,n}$, if necessary, i.e., in the case that $l \neq 0$ and $n \neq 0$) and the corresponding rotation angle $l\alpha \pm n\beta$ have been presented in Table 1 for all $(2k, l, 2m, n)$ combinations up to fourth order, i.e., $2k+l+2m+n \leq 4$.

Relationships between the moments in the fractional (α, β) domain and the moments in the original domain follow from the relations [cf. Eq. (21)]

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x, u; y, v)\xi_{2k,l}\eta_{2m,n}dxdudydv \\ = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x, u; y, v)\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)dxdudydv \\ = &\exp[-j(l\alpha + n\beta)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x, u; y, v)\xi_{2k,l}\eta_{2m,n}dxdudydv \end{aligned} \tag{34}$$

and, if necessary,

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x, u; y, v)\xi_{2k,l}\eta^*_{2m,n}dxdudydv \\ = &\exp[-j(l\alpha - n\beta)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x, u; y, v)\xi_{2k,l}\eta^*_{2m,n}dxdudydv, \end{aligned} \tag{35}$$

together with the expressions given in Table 1, which results in rotation-type relationships of the form of Eq. (32). The last row in Table 1, for instance, leads to the relationship

$$\begin{bmatrix} \mu_{0040}(\alpha,\beta) - 6\mu_{0022}(\alpha,\beta) + \mu_{0004}(\alpha,\beta) \\ 4\mu_{0031}(\alpha,\beta) - 4\mu_{0013}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(4\beta) \begin{bmatrix} \mu_{0040} - 6\mu_{0022} + \mu_{0004} \\ 4\mu_{0031} - 4\mu_{0013} \end{bmatrix}.$$

The two vector entries (top entry t and bottom entry b) and the corresponding rotation angle $l\alpha \pm n\beta$ for all different cases (up to fourth order) have been given in Table 2.

A. First-order moments

For the first-order moments we get 2 sets of equations, cf. Table 2: 2 equations for the x and u moments (with the moment indices $r = s = 0$),

$$\begin{bmatrix} \mu_{1000}(\alpha,\beta) \\ \mu_{0100}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\alpha) \begin{bmatrix} \mu_{1000} \\ \mu_{0100} \end{bmatrix} \tag{36}$$

and 2 equations for the y and v moments (with the moment indices $p = q = 0$),

$$\begin{bmatrix} \mu_{0010}(\alpha,\beta) \\ \mu_{0001}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\beta) \begin{bmatrix} \mu_{0010} \\ \mu_{0001} \end{bmatrix}, \tag{37}$$

which constitute 4 equations for the 4 first-order moments. From the first rows of the two vector-matrix equations (36) and (37) we get

$$\mu_{1000}(\alpha,\beta) = \mu_{1000}\cos\alpha + \mu_{0100}\sin\alpha, \tag{38}$$

$$\mu_{0010}(\alpha,\beta) = \mu_{0010}\cos\beta + \mu_{0001}\sin\beta, \tag{39}$$

which equations correspond to Eq. (23) with $pqrs = 1000$ and $pqrs = 0010$, respectively, and the 4 moments μ_{1000}, μ_{0100}, μ_{0010}, and μ_{0001} can be determined by measuring the intensity moments $\mu_{1000}(\alpha,\cdot)$ and $\mu_{0010}(\cdot,\beta)$ in the fractional domain for two values of α and β, for instance for 0 and $\frac{1}{2}\pi$: $\mu_{1000} = \mu_{1000}(0,\cdot)$, $\mu_{0100} = \mu_{1000}(\frac{1}{2}\pi,\cdot)$, $\mu_{0010} = \mu_{0010}(\cdot,0)$, and $\mu_{0001} = \mu_{0010}(\cdot,\frac{1}{2}\pi)$.

Note that the following two expressions

$$\mu_{1000}^2 + \mu_{0100}^2, \tag{40}$$

$$\mu_{0010}^2 + \mu_{0001}^2, \tag{41}$$

are invariant under fractional Fourier transformation.

B. Second-order moments

For the second-order moments we get 3 sets of equations, cf. Table 2: 3 equations for the x and u moments (with the indices $r = s = 0$ again),

$$\begin{aligned} \mu_{2000}(\alpha,\beta) + \mu_{0200}(\alpha,\beta) &= \mu_{2000} + \mu_{0200}, \\ \begin{bmatrix} \mu_{2000}(\alpha,\beta) - \mu_{0200}(\alpha,\beta) \\ 2\mu_{1100}(\alpha,\beta) \end{bmatrix} &= \mathbf{R}(2\alpha) \begin{bmatrix} \mu_{2000} - \mu_{0200} \\ 2\mu_{1100} \end{bmatrix}, \end{aligned} \tag{42}$$

4 equations for the mixed moments,

$$\begin{aligned} \begin{bmatrix} \mu_{1010}(\alpha,\beta) - \mu_{0101}(\alpha,\beta) \\ \mu_{1001}(\alpha,\beta) + \mu_{0110}(\alpha,\beta) \end{bmatrix} &= \mathbf{R}(\alpha+\beta) \begin{bmatrix} \mu_{1010} - \mu_{0101} \\ \mu_{1001} + \mu_{0110} \end{bmatrix}, \\ \begin{bmatrix} \mu_{1010}(\alpha,\beta) + \mu_{0101}(\alpha,\beta) \\ -\mu_{1001}(\alpha,\beta) + \mu_{0110}(\alpha,\beta) \end{bmatrix} &= \mathbf{R}(\alpha-\beta) \begin{bmatrix} \mu_{1010} + m_{0101} \\ -\mu_{1001} + m_{0110} \end{bmatrix}, \end{aligned} \tag{43}$$

and 3 equations for the y and v moments (with the indices $p = q = 0$ again),

$$\begin{aligned} \mu_{0020}(\alpha,\beta) + \mu_{0002}(\alpha,\beta) &= \mu_{0020} + \mu_{0002}, \\ \begin{bmatrix} \mu_{0020}(\alpha,\beta) - \mu_{0002}(\alpha,\beta) \\ 2\mu_{0011}(\alpha,\beta) \end{bmatrix} &= \mathbf{R}(2\beta) \begin{bmatrix} \mu_{0020} - \mu_{0002} \\ 2\mu_{0011} \end{bmatrix}, \end{aligned} \tag{44}$$

which constitute 3+4+3=10 equations for the 10 second-order moments. From the first rows of the vector-matrix equations – or directly from Eq. (23) with $pqrs = 2000$, $pqrs = 1010$, and $pqrs = 0020$, respectively – we derive the 3 relationships

$$\mu_{2000}(\alpha,\beta) = \mu_{2000}\cos^2\alpha + 2\mu_{1100}\cos\alpha\sin\alpha + \mu_{0200}\sin^2\alpha, \tag{45}$$

$$\mu_{1010}(\alpha,\beta) = \mu_{1010}\cos\alpha\cos\beta + \mu_{1001}\cos\alpha\sin\beta + \mu_{0110}\sin\alpha\cos\beta + \mu_{0101}\sin\alpha\sin\beta, \tag{46}$$

$$\mu_{0020}(\alpha,\beta) = \mu_{0020}\cos^2\beta + 2\mu_{0011}\cos\beta\sin\beta + \mu_{0002}\sin^2\beta. \tag{47}$$

The 3 moments μ_{2000}, μ_{1100}, and μ_{0200} can be determined by measuring the intensity moment $\mu_{2000}(\alpha,\cdot)$ in the fractional domain[26] for three values of α, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{2000}=\mu_{2000}(0,\cdot)$, $\mu_{0200}=\mu_{2000}(\frac{1}{2}\pi,\cdot)$, and then $\mu_{1100}=\mu_{2000}(\frac{1}{4}\pi,\cdot)-\frac{1}{2}(\mu_{2000}+\mu_{0200})$. Likewise, the 3 moments μ_{0020}, μ_{0011}, and μ_{0002} can be determined by measuring the intensity moment $\mu_{0020}(\cdot,\beta)$ for three values of β, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{0020}=\mu_{0020}(\cdot,0)$, $\mu_{0002}=\mu_{0020}(\cdot,\frac{1}{2}\pi)$, and then $\mu_{0011}=\mu_{0020}(\cdot,\frac{1}{4}\pi)-\frac{1}{2}(\mu_{0020}+\mu_{0002})$. The other 4 moments μ_{1010}, μ_{1001}, μ_{0110}, and μ_{0101} follow from measuring the intensity moment $\mu_{1010}(\alpha,\beta)$, for instance as follows: $\mu_{1010}=\mu_{1010}(0,0)$, $\mu_{0101}=\mu_{1010}(\frac{1}{2}\pi,\frac{1}{2}\pi)$, $\mu_{0110}=\mu_{1010}(\frac{1}{2}\pi,0)$, and then $\mu_{1001}=2\mu_{1010}(\frac{1}{4}\pi,\frac{1}{4}\pi)-\mu_{1010}-\mu_{0110}-\mu_{0101}$. We conclude that all 10 second-order moments can be determined from the knowledge of 4 fractional power spectra, where one of them has to be anamorphic (i.e., $\alpha\neq\beta$), for instance $|F_{0,0}(x,y)|^2$, $|F_{\pi/4,\pi/4}(x,y)|^2$, $|F_{\pi/2,\pi/2}(x,y)|^2$, and $|F_{\pi/2,0}(x,y)|^2$.

Note that we have the following 2+2+2=6 invariants, which can directly be derived from Table 2 as invariants of t^2+b^2:

$$\begin{aligned}
\mu_{2000} &+ \mu_{0200}, && (48)\\
(\mu_{2000}-\mu_{0200})^2 &+ 4\mu_{1100}^2, && (49)\\
(\mu_{1010}-\mu_{0101})^2 &+ (\mu_{1001}+\mu_{0110})^2, && (50)\\
(\mu_{1010}+\mu_{0101})^2 &+ (-\mu_{1001}+\mu_{0110})^2, && (51)\\
\mu_{0020} &+ \mu_{0002}, && (52)\\
(\mu_{0020}-\mu_{0002})^2 &+ 4\mu_{0011}^2. && (53)
\end{aligned}$$

Combinations of these expressions lead to more invariants, like $\mu_{2000}^2+2\mu_{1100}^2+\mu_{0200}^2$ and $\mu_{2000}\mu_{0200}-\mu_{1100}^2$ for the x and u moments [with similar expressions for the y and v moments μ_{0020}, μ_{0011}, and μ_{0002}], and $\mu_{1010}^2+\mu_{1001}^2+\mu_{0110}^2+\mu_{0101}^2$ and $\mu_{1010}\mu_{0101}-\mu_{1001}\mu_{0110}$ for the mixed moments.

From De Bruijn's inequality for the WD of a one-dimensional function,[27]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(c^2x^2+\frac{u^2}{c^2}\right)^m W_f(x,u)dxdu \geq m!\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u)dxdu \tag{54}$$

and choosing $c=1$, we conclude that the invariants (48) and (52) have a lower bound of 1. Choosing $c^2=\sqrt{\mu_{0200}/\mu_{2000}}$ in De Bruijn's inequality leads to the well-known uncertainty relation $2\sqrt{\mu_{2000}\mu_{0200}}\geq 1$, and a similar expression holds for the moments μ_{2000} and μ_{0200}.

C. Higher-order moments

For higher-order moments we can proceed analogously, using the expressions given in Table 2. For the third-order moments we get 4 sets of equations, which yield 4+6+6+4=20 equations for 20 variables. From these equations we get 4 relationships from which the 20 third-order moments can be determined:

$$\begin{aligned}
\mu_{3000}(\alpha,\beta) &= \mu_{3000}\cos^3\alpha+3\mu_{2100}\cos^2\alpha\sin\alpha+3\mu_{1200}\cos\alpha\sin^2\alpha+\mu_{0300}\sin^3\alpha, && (55)\\
\mu_{2010}(\alpha,\beta) &= \mu_{2010}\cos^2\alpha\cos\beta+\mu_{2001}\cos^2\alpha\sin\beta+2\mu_{1110}\cos\alpha\sin\alpha\cos\beta\\
&\quad +2\mu_{1101}\cos\alpha\sin\alpha\sin\beta+\mu_{0210}\sin^2\alpha\cos\beta+\mu_{0201}\sin^2\alpha\sin\beta, && (56)\\
\mu_{1020}(\alpha,\beta) &= \mu_{1020}\cos\alpha\cos^2\beta+2\mu_{1011}\cos\alpha\cos\beta\sin\beta+\mu_{1002}\cos\alpha\sin^2\beta\\
&\quad +\mu_{0120}\sin\alpha\cos^2\beta+2\mu_{0111}\sin\alpha\cos\beta\sin\beta+\mu_{0102}\sin\alpha\sin^2\beta, && (57)\\
\mu_{0030}(\alpha,\beta) &= \mu_{0030}\cos^3\beta+3\mu_{0021}\cos^2\beta\sin\beta+3\mu_{0012}\cos\beta\sin^2\beta+\mu_{0003}\sin^3\beta. && (58)
\end{aligned}$$

Note again that these relations could as well have been derived directly from Eq. (23) with $pqrs=3000$, $pqrs=2010$, $pqrs=1020$, and $pqrs=0030$, respectively. The 20 third-order moments can be determined from the knowledge of 6 fractional power spectra, where 2 of them have to be anamorphic.

We remark throughout the correspondence between the moment indices p, q, r, and s on the one hand, and the powers of $\cos\alpha$, $\sin\alpha$, $\cos\beta$, and $\sin\beta$ on the other hand, respectively. Moreover, the numerical factor in front of the moments corresponds to the number of variables permutations that lead to identical moments; for instance, the moment μ_{2100}, stems from three possible permutations $xxu = xux = uxx$. This observation leads to an easy way [cf. Eq. (23)] to find equations of the above type, without using Table 2. To construct Eq. (56), for example, we would proceed as follows. We first note that the moment $\mu_{2010}(\alpha, \beta)$ in the left-hand side stems from xxy. The moments that will appear in the right-hand side then stem from xxy (leading to $\mu_{2010}\cos^2\alpha\cos\beta$) and from all the variables combinations that result from interchanging x with u, and y with v: xxv (leading to $\mu_{2001}\cos^2\alpha\sin\beta$), $xuy = uxy$ (leading to $2\mu_{1110}\cos\alpha\sin\alpha\cos\beta$), $xuv = uxv$ (leading to $2\mu_{1101}\cos\alpha\sin\alpha\sin\beta$), uuy (leading to $\mu_{0210}\sin^2\alpha\cos\beta$) and uuv (leading to $\mu_{0201}\sin^2\alpha\sin\beta$). This procedure can, of course, be applied to higher-order moments, as well.

For the fourth-order moments we get 5 sets of equations, which yield 5+8+9+8+5=35 equations for 35 variables. From these equations we can derive 5 relationships from which the 35 fourth-order moments can be determined [cf. Eq. (23) with $pqrs = 4000$, $pqrs = 3010$, $pqrs = 2020$, $pqrs = 1030$, and $pqrs = 0040$, respectively]:

$$\begin{aligned}
\mu_{4000}(\alpha,\beta) &= \mu_{4000}\cos^4\alpha + 4\mu_{3100}\cos^3\alpha\sin\alpha + 6\mu_{2200}\cos^2\alpha\sin^2\alpha \\
&\quad + 4\mu_{1300}\cos\alpha\sin^3\alpha + \mu_{0400}\sin^4\alpha, \qquad (59)\\
\mu_{3010}(\alpha,\beta) &= \mu_{3010}\cos^3\alpha\cos\beta + \mu_{3001}\cos^3\alpha\sin\beta + 3\mu_{2110}\cos^2\alpha\sin\alpha\cos\beta \\
&\quad + 3\mu_{2101}\cos^2\alpha\sin\alpha\sin\beta + 3\mu_{1210}\cos\alpha\sin^2\alpha\cos\beta + 3\mu_{1201}\cos\alpha\sin^2\alpha\sin\beta \\
&\quad + \mu_{0310}\sin^3\alpha\cos\beta + \mu_{0301}\sin^3\alpha\sin\beta, \qquad (60)\\
\mu_{2020}(\alpha,\beta) &= \mu_{2020}\cos^2\alpha\cos^2\beta + 2\mu_{2011}\cos^2\alpha\cos\beta\sin\beta + \mu_{2002}\cos^2\alpha\sin^2\beta \\
&\quad + 2\mu_{1120}\cos\alpha\sin\alpha\cos^2\beta + 4\mu_{1111}\cos\alpha\sin\alpha\cos\beta\sin\beta \\
&\quad + 2\mu_{1102}\cos\alpha\sin\alpha\sin^2\beta + \mu_{0220}\sin^2\alpha\cos^2\beta + 2\mu_{0211}\sin^2\alpha\cos\beta\sin\beta \\
&\quad + \mu_{0202}\sin^2\alpha\sin^2\beta, \qquad (61)\\
\mu_{1030}(\alpha,\beta) &= \mu_{1030}\cos\alpha\cos^3\beta + 3\mu_{1021}\cos\alpha\cos^2\beta\sin\beta + 3\mu_{1012}\cos\alpha\cos\beta\sin^2\beta \\
&\quad + \mu_{1003}\cos\alpha\sin^3\beta + \mu_{0130}\sin\alpha\cos^3\beta + 3\mu_{0121}\sin\alpha\cos^2\beta\sin\beta \\
&\quad + 3\mu_{0112}\sin\alpha\cos\beta\sin^2\beta + \mu_{0103}\sin\alpha\sin^3\beta, \qquad (62)\\
\mu_{0040}(\alpha,\beta) &= \mu_{0040}\cos^4\beta + 4\mu_{0031}\cos^3\beta\sin\beta + 6\mu_{0022}\cos^2\beta\sin^2\beta \\
&\quad + 4\mu_{0013}\cos\beta\sin^3\beta + \mu_{0004}\sin^4\beta. \qquad (63)
\end{aligned}$$

The 35 fourth-order moments can be determined from the knowledge of 9 fractional power spectra, where 4 of them have to be anamorphic.

The 2+3+3+2=10 invariant combinations of third-order moments and the 3+4+5+4+3=19 invariant combinations of fourth-order moments follow directly from Table 2 as invariants of $t^2 + b^2$. We mention three simple invariants for fourth-order moments in particular: the combination

$$\mu_{4000} + 2\mu_{2200} + \mu_{0400}, \qquad (64)$$

related to x^2 and u^2 (with the indices $r = s = 0$), cf. the second-order invariant $\mu^2_{2000} + 2\mu^2_{1100} + \mu^2_{0200}$; the similar combination

$$\mu_{0040} + 2\mu_{0022} + \mu_{0004}, \qquad (65)$$

related to y^2 and v^2 (with the indices $p = q = 0$), cf. the second-order invariant $\mu^2_{0020} + 2\mu^2_{0011} + \mu^2_{0002}$; and the combination

$$\mu_{2020} + \mu_{2002} + \mu_{0220} + \mu_{0202}, \qquad (66)$$

where the squares of all four variables x^2, u^2, y^2, and v^2 enter the expression, cf. the second-order invariant $\mu_{1010}^2 + \mu_{1001}^2 + \mu_{0110}^2 + \mu_{0101}^2$.

Note again that De Bruijn's inequality (54) leads to a lower bound of 2 for the expressions (64) and (65); moreover, the choice $c^4 = \sqrt{\mu_{0400}/\mu_{4000}}$ leads to the interesting inequality $\sqrt{\mu_{4000}\mu_{0400}} + \mu_{2200} \geq 1$, and a similar expression holds for the moments μ_{0040}, μ_{0004}, and μ_{0022}.

To find the number of nth-order moments N, and the total number of fractional power spectra N_t (with N_a the number of anamorphic ones) that we need to determine these N moments, use can be made of the following triangle, which can easily be extended to higher order:

n	number of nth order moments	N	N_t	N_a
0	1	1	1	0
1	$2+2$	4	2	0
2	$3+4+3$	10	4	1
3	$4+6+6+4$	20	6	2
4	$5+8+9+8+5$	35	9	4
5	$6+10+12+12+10+6$	56	12	6
6	$7+12+15+16+15+12+7$	84	16	9
7	$8+14+18+20+20+18+14+8$	120	20	12
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that N (the number of nth-order moments, with $n = p + q + r + s$) is equal to the sum of the values in the nth row of the triangle, $N = \frac{1}{6}(n+1)(n+2)(n+3)$; that N_t (the total number of fractional power spectra) is equal to the highest value that appears in the nth row of the triangle, $N_t = \frac{1}{4}(n+2)^2$ for $n =$ even, and $N_t = \frac{1}{4}(n+3)(n+1)$ for $n =$ odd; and that N_a (the number of anamorphic fractional power spectra) follows from $N_a = N_t - (n+1)$.

6. Conclusions

We have shown how all global WD moments of arbitrary order (or derivatives of the AF at the origin) can be measured as intensity moments in the output plane of an appropriate number of fractional FT systems (generally anamorphic ones, i.e., with different angles α and β), and we have derived the minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments. The results followed directly from the general relationship (23) that expresses the intensity moments in the output plane of an anamorphic fractional FT system in terms of the moments in the input plane and the two angles α and β. They could also be derived by formulating rotation-type input-output-relationships between certain combinations of moments. The latter method yields, as a by-product, a number of moment combinations that are invariant under anamorphic fractional Fourier transformation.

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York, 1967), pp. 57–71.

Table 1. Real and imaginary parts of $\xi_{2k,l}\eta_{2m,n}$ (and $\xi_{2k,l}\eta^*_{2m,n}$, if necessary) up to fourth order

$\xi_{2k,l}\eta_{2m,n}$	$\Re\{\xi_{2k,l}\eta_{2m,n}\}$	$\Im\{\xi_{2k,l}\eta_{2m,n}\}$	$l\alpha \pm n\beta$
$\xi_{01}\eta_{00}$	x	u	α
$\xi_{00}\eta_{01}$	y	v	β
$\xi_{20}\eta_{00}$	x^2+u^2		
$\xi_{02}\eta_{00}$	x^2-u^2	$2xu$	2α
$\xi_{01}\eta_{01}$	$xy-uv$	$xv+uy$	$\alpha+\beta$
$\xi_{01}\eta^*_{01}$	$xy+uv$	$-xv+uy$	$\alpha-\beta$
$\xi_{00}\eta_{20}$	y^2+v^2		
$\xi_{00}\eta_{02}$	y^2-v^2	$2yv$	2β
$\xi_{21}\eta_{00}$	x^3+xu^2	x^2u+u^3	α
$\xi_{03}\eta_{00}$	x^3-3xu^2	$3x^2u-u^3$	3α
$\xi_{20}\eta_{01}$	x^2y+u^2y	x^2v+u^2v	β
$\xi_{02}\eta_{01}$	$x^2y-u^2y-2xuv$	$x^2v-u^2v+2xuy$	$2\alpha+\beta$
$\xi_{02}\eta^*_{01}$	$x^2y-u^2y+2xuv$	$-x^2v+u^2v+2xuy$	$2\alpha-\beta$
$\xi_{01}\eta_{20}$	xy^2+xv^2	uy^2+uv^2	α
$\xi_{01}\eta_{02}$	xy^2-xv^2-2uyv	$2xyv+uy^2-uv^2$	$\alpha+2\beta$
$\xi_{01}\eta^*_{02}$	xy^2-xv^2+2uyv	$-2xyv+uy^2-uv^2$	$\alpha-2\beta$
$\xi_{00}\eta_{21}$	y^3+yv^2	y^2v+v^3	β
$\xi_{00}\eta_{03}$	y^3-3yv^2	$3y^2v-v^3$	3β
$\xi_{40}\eta_{00}$	$x^4+2x^2u^2+u^4$		
$\xi_{22}\eta_{00}$	x^4-u^4	$2x^3u+2xu^3$	2α
$\xi_{04}\eta_{00}$	$x^4-6x^2u^2+u^4$	$4x^3u-4xu^3$	4α
$\xi_{21}\eta_{01}$	$x^3y+xu^2y-x^2uv-u^3v$	$x^3v+xu^2v+x^2uy+u^3y$	$\alpha+\beta$
$\xi_{21}\eta^*_{01}$	$x^3y+xu^2y+x^2uv+u^3v$	$-x^3v-xu^2v+x^2uy+u^3y$	$\alpha-\beta$
$\xi_{03}\eta_{01}$	$x^3y-3xu^2y-3x^2uv+u^3v$	$x^3v-3xu^2v+3x^2uy-u^3y$	$3\alpha+\beta$
$\xi_{03}\eta^*_{01}$	$x^3y-3xu^2y+3x^2uv-u^3v$	$-x^3v+3xu^2v+3x^2uy-u^3y$	$3\alpha-\beta$
$\xi_{20}\eta_{20}$	$x^2y^2+x^2v^2+u^2y^2+u^2v^2$		
$\xi_{20}\eta_{02}$	$x^2y^2-x^2v^2+u^2y^2-u^2v^2$	$2x^2yv+2u^2yv$	2β
$\xi_{02}\eta_{20}$	$x^2y^2+x^2v^2-u^2y^2-u^2v^2$	$2xuy^2+2xuv^2$	2α
$\xi_{02}\eta_{02}$	$x^2y^2-x^2v^2-u^2y^2+u^2v^2-4xuyv$	$2x^2yv-2u^2yv+2xuy^2-2xuv^2$	$2\alpha+2\beta$
$\xi_{02}\eta^*_{02}$	$x^2y^2-x^2v^2-u^2y^2+u^2v^2+4xuyv$	$-2x^2yv+2u^2yv+2xuy^2-2xuv^2$	$2\alpha-2\beta$
$\xi_{01}\eta_{21}$	$xy^3+xyv^2-uy^2v-uv^3$	$xy^2v+xv^3+uy^3+uyv^2$	$\alpha+\beta$
$\xi_{01}\eta^*_{21}$	$xy^3+xyv^2+uy^2v+uv^3$	$-xy^2v-xv^3+uy^3+uyv^2$	$\alpha-\beta$
$\xi_{01}\eta_{03}$	$xy^3-3xyv^2-3uy^2v+uv^3$	$3xy^2v-xv^3+uy^3-3uyv^2$	$\alpha+3\beta$
$\xi_{01}\eta^*_{03}$	$xy^3-3xyv^2+3uy^2v-uv^3$	$-3xy^2v+xv^3+uy^3-3uyv^2$	$\alpha-3\beta$
$\xi_{00}\eta_{40}$	$y^4+2y^2v^2+v^4$		
$\xi_{00}\eta_{22}$	y^4-v^4	$2y^3v+2yv^3$	2β
$\xi_{00}\eta_{04}$	$y^4-6y^2v^2+v^4$	$4y^3v-4yv^3$	4β

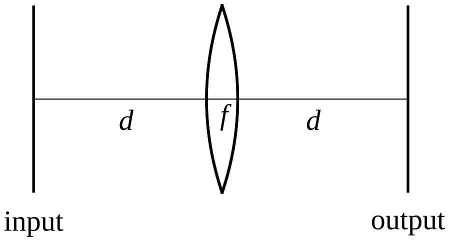
Table 2. Moment combinations undergoing a rotation of the form of Eqs. (32) up to fourth order

$$\begin{bmatrix} t(\alpha,\beta) \\ b(\alpha,\beta) \end{bmatrix} = \mathbf{R}(l\alpha \pm n\beta) \begin{bmatrix} t(0,0) \\ b(0,0) \end{bmatrix} = \begin{bmatrix} \cos(l\alpha \pm n\beta) & \sin(l\alpha \pm n\beta) \\ -\sin(l\alpha \pm n\beta) & \cos(l\alpha \pm n\beta) \end{bmatrix} \begin{bmatrix} t(0,0) \\ b(0,0) \end{bmatrix}$$

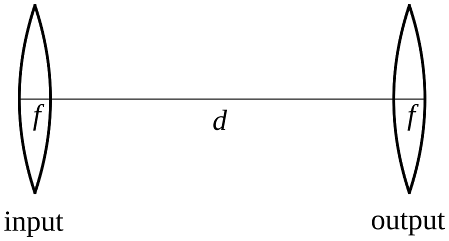
top vector entry t	bottom vector entry b	angle $l\alpha \pm n\beta$
μ_{1000}	μ_{0100}	α
μ_{0010}	μ_{0001}	β
$\mu_{2000} + \mu_{0200}$		
$\mu_{2000} - \mu_{0200}$	$2\mu_{1100}$	2α
$\mu_{1010} - \mu_{0101}$	$\mu_{1001} + \mu_{0110}$	$\alpha + \beta$
$\mu_{1010} + \mu_{0101}$	$-\mu_{1001} + \mu_{0110}$	$\alpha - \beta$
$\mu_{0020} + \mu_{0002}$		
$\mu_{0020} - \mu_{0002}$	$2\mu_{0011}$	2β
$\mu_{3000} + \mu_{1200}$	$\mu_{2100} + \mu_{0300}$	α
$\mu_{3000} - 3\mu_{1200}$	$3\mu_{2100} - \mu_{0300}$	3α
$\mu_{2010} + \mu_{0210}$	$\mu_{2001} + \mu_{0201}$	β
$\mu_{2010} - \mu_{0210} - 2\mu_{1101}$	$\mu_{2001} - \mu_{0201} + 2\mu_{1110}$	$2\alpha + \beta$
$\mu_{2010} - \mu_{0210} + 2\mu_{1101}$	$-\mu_{2001} + \mu_{0201} + 2\mu_{1110}$	$2\alpha - \beta$
$\mu_{1020} + \mu_{1002}$	$\mu_{0120} + \mu_{0102}$	α
$\mu_{1020} - \mu_{1002} - 2\mu_{0111}$	$2\mu_{1011} + \mu_{0120} - \mu_{0102}$	$\alpha + 2\beta$
$\mu_{1020} - \mu_{1002} + 2\mu_{0111}$	$-2\mu_{1011} + \mu_{0120} - \mu_{0102}$	$\alpha - 2\beta$
$\mu_{0030} + \mu_{0012}$	$\mu_{0021} + \mu_{0003}$	β
$\mu_{0030} - 3\mu_{0012}$	$3\mu_{0021} - \mu_{0003}$	3β
$\mu_{4000} + 2\mu_{2200} + \mu_{0400}$		
$\mu_{4000} - \mu_{0400}$	$2\mu_{3100} + 2\mu_{1300}$	2α
$\mu_{4000} - 6\mu_{2200} + \mu_{0400}$	$4\mu_{3100} - 4\mu_{1300}$	4α
$\mu_{3010} + \mu_{1210} - \mu_{2101} - \mu_{0301}$	$\mu_{3001} + \mu_{1201} + \mu_{2110} + \mu_{0310}$	$\alpha + \beta$
$\mu_{3010} + \mu_{1210} + \mu_{2101} + \mu_{0301}$	$-\mu_{3001} - \mu_{1201} + \mu_{2110} + \mu_{0310}$	$\alpha - \beta$
$\mu_{3010} - 3\mu_{1210} - 3\mu_{2101} + \mu_{0301}$	$\mu_{3001} - 3\mu_{1201} + 3\mu_{2110} - \mu_{0310}$	$3\alpha + \beta$
$\mu_{3010} - 3\mu_{1210} + 3\mu_{2101} - \mu_{0301}$	$-\mu_{3001} + 3\mu_{1201} + 3\mu_{2110} - \mu_{0310}$	$3\alpha - \beta$
$\mu_{2020} + \mu_{2002} + \mu_{0220} + \mu_{0202}$		
$\mu_{2020} - \mu_{2002} + \mu_{0220} - \mu_{0202}$	$2\mu_{2011} + 2\mu_{0211}$	2β
$\mu_{2020} + \mu_{2002} - \mu_{0220} - \mu_{0202}$	$2\mu_{2120} + 2\mu_{1102}$	2α
$\mu_{2020} - \mu_{2002} - \mu_{0220} + \mu_{0202} - 4\mu_{1111}$	$2\mu_{2011} - 2\mu_{0211} + 2\mu_{1120} - 2\mu_{1102}$	$2\alpha + 2\beta$
$\mu_{2020} - \mu_{2002} - \mu_{0220} + \mu_{0202} + 4\mu_{1111}$	$-2\mu_{2011} + 2\mu_{0211} + 2\mu_{1120} - 2\mu_{1102}$	$2\alpha - 2\beta$
$\mu_{1030} + \mu_{1012} - \mu_{0121} - \mu_{0103}$	$\mu_{1201} + \mu_{1003} + \mu_{0130} + \mu_{0112}$	$\alpha + \beta$
$\mu_{1030} + \mu_{1012} + \mu_{0121} + \mu_{0103}$	$-\mu_{1201} - \mu_{1003} + \mu_{0130} + \mu_{0112}$	$\alpha - \beta$
$\mu_{1030} - 3\mu_{1012} - 3\mu_{0121} + \mu_{0103}$	$3\mu_{1201} - \mu_{1003} + \mu_{0130} - 3\mu_{0112}$	$\alpha + 3\beta$
$\mu_{1030} - 3\mu_{1012} + 3\mu_{0121} - \mu_{0103}$	$-3\mu_{1201} + \mu_{1003} + \mu_{0130} - 3\mu_{0112}$	$\alpha - 3\beta$
$\mu_{0040} + 2\mu_{0022} + \mu_{0004}$		
$\mu_{0040} - \mu_{0004}$	$2\mu_{0031} + 2\mu_{0013}$	2β
$\mu_{0040} - 6\mu_{0022} + \mu_{0004}$	$4\mu_{0031} - 4\mu_{0013}$	4β

Figure caption

Figure 1.
Two simple coherent-optical fractional FT systems,[20] whose point-spread functions take the form of Eq. (14), (a) using one thin (cylindrical) lens with focal length f, preceded and followed by two identical distances d of free space, and (b) using two identical thin (cylindrical) lenses with focal lengths f, separated by a distance d. The relation between d, f, and the fractional angle α reads $d = 2f \sin^2 \frac{1}{2}\alpha$ in both set-ups; the 'real' space coordinates in the input and output planes are proportional to x and u, respectively, by the proportionality factor $(\lambda f \sin\alpha)^{1/2}$ in set-up (a) and $(\lambda f \tan \frac{1}{2}\alpha)^{1/2}$ in set-up (b), where λ is the wavelength of the laser light.



(a)



(b)